GW
GATEWAY CLASSES

# AKTU B.TECH SEM-III

## EE & EE ALLIED

# EMFT

## ( ELECTROMAGNETIC FIELD THEORY )

### Unit-1 Lec--1

**Topics to be covered**

- ➢ Basics of Vectors
  - Addition
  - Subtraction
  - Multiplication

PYQs

**By Swati Ma'am**

# Udaan Batch 2.0
## B.Tech II Year
## EE & EE Allied

## Electromagnetic Field Theory (BEE301)

### Unit-1 Syllabus

**Coordinate Systems and Transformation:** Basics of Vectors: Addition, subtraction and multiplications; Cartesian, Cylindrical, Spherical transformation. Vector calculus: Differential length, area and volume, line surface and volume integrals. Del operator, Gradient, Divergence of a vector, Divergence theorem, Curl of a vector, Stokes's theorem, Laplacian of a scalar.

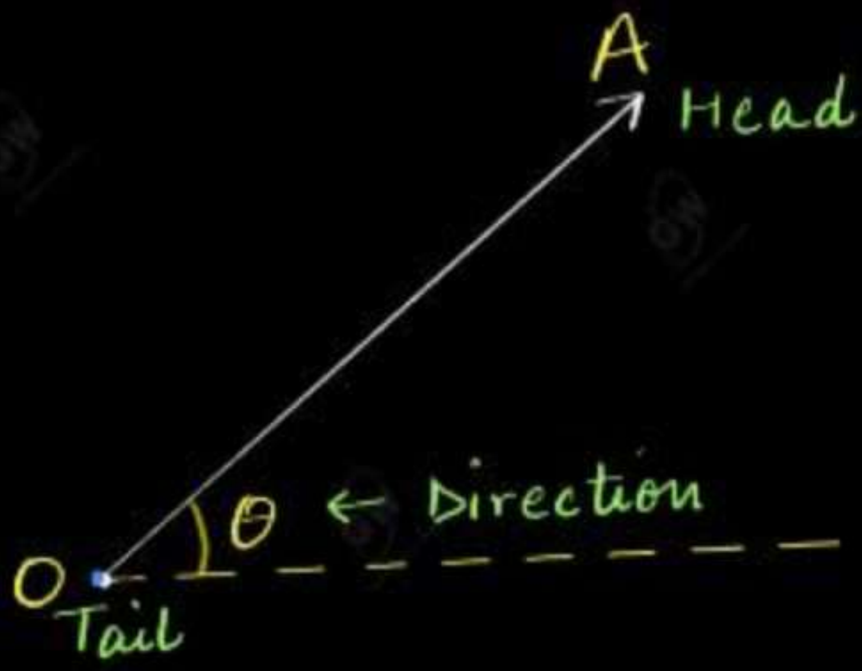
# VECTOR CALCULUS

**Scalar Quantities:-** Quantities which have magnitude but no direction.

**Examples :–** Mass, Time, Length, Temperature, Energy, Work, Pressure etc

**Vector Quantities:-** Quantities which have magnitude as well as direction.

**Examples :–** Electric field, Magnetic Field, Velocity, Acceleration, Force etc.



$\vec{OA}$ or $\vec{A}$

Magnitude of $\vec{A}$ is $|\vec{A}|$

# UNIT VECTOR

**Unit Vector:-** is a vector that has a magnitude of exactly 1 and is drawn in the direction of the given vector.

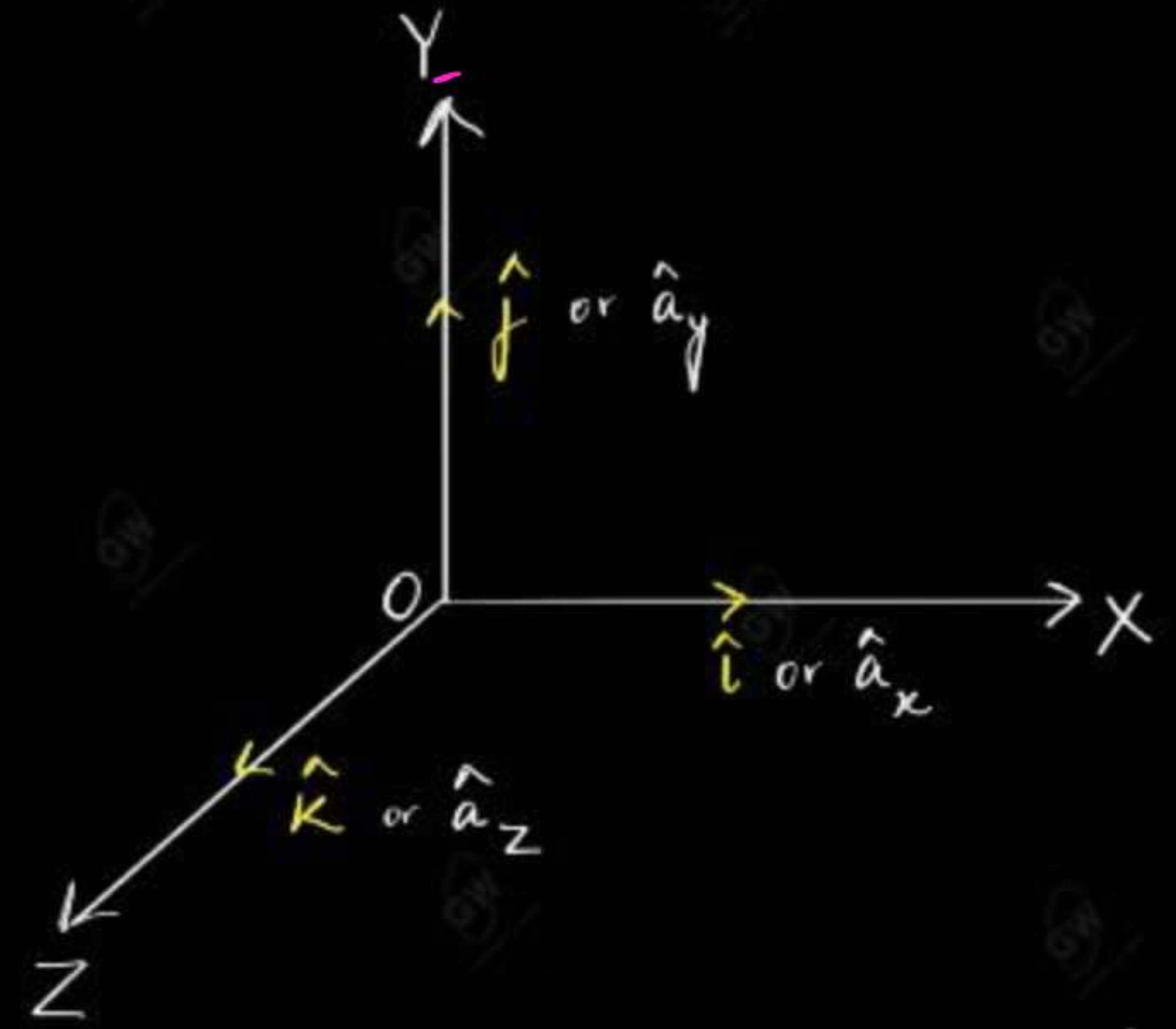
**Unit Vector = [Vector] / [Vector's magnitude]**

$$\boxed{\hat{A} = \frac{\vec{A}}{|\vec{A}|}}$$

Vector = [Vector's magnitude] · Unit Vector

$$\vec{A} = |\vec{A}| \cdot \hat{A} = A \cdot \hat{A}$$

Vector = Scalar · Unit Vector


Y
$\hat{j}$ or $\hat{a}_y$
O
X
$\hat{i}$ or $\hat{a}_x$
$\hat{k}$ or $\hat{a}_z$
Z

$\hat{i}$ or $\hat{a}_x$ :- Unit vector along X axis

$\hat{j}$ or $\hat{a}_y$ :- Unit vector along Y axis

$\hat{k}$ or $\hat{a}_z$ :- Unit vector along Z axis

**Que-1) Find the unit vector which is in the direction of $3\hat{i} + 4\hat{j} - 5\hat{k}$.**

**Solution:** Given vector A = $3\hat{i} + 4\hat{j} - 5\hat{k}$

Its magnitude is, $|A| = \sqrt{(3^2 + 4^2 + (-5)^2)}$

$= \sqrt{(9 + 16 + 25)}$

$= \sqrt{50}$

$= 5\sqrt{2}$

The unit vector in the direction of the given vector is $\vec{A}/|\vec{A}|$

$= (3\hat{i} + 4\hat{j} - 5\hat{k}) / (5\sqrt{2})$ $\quad = \dfrac{3\hat{i} + 4\hat{j} - 5\hat{k}}{5\sqrt{2}}$

**Hence the unit vector is $(3/5\sqrt{2})\,\hat{i} + (4/5\sqrt{2})\,\hat{j} - (5/5\sqrt{2})\,\hat{k}$.** $\quad = \dfrac{3}{5\sqrt{2}}\hat{i} + \dfrac{4}{5\sqrt{2}}\hat{j} - \dfrac{5}{5\sqrt{2}}\hat{k}$ Ans.

**Que-2) Find the unit vector extending from the origin toward the point G (2, −2, −1).**

$$\hat{G} = \frac{\vec{G}}{|\vec{G}|}$$

$$\vec{G} = 2\hat{i} - 2\hat{j} - \hat{k}$$

$$|\vec{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2}$$

$$= \sqrt{4+4+1}$$

$$= \sqrt{9}$$

$$= 3$$

$$\therefore \hat{G} = \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3}$$

$$= \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

$$= 0.667\hat{i} - 0.667\hat{j} - 0.333\hat{k}$$

Ans.

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Find the unit vector of the vector

$$\vec{A} = 7\hat{a}_x + 2\hat{a}_y + 8\hat{a}_z$$

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

$$|\vec{A}| = \sqrt{(7)^2 + (2)^2 + (8)^2}$$

$$= \sqrt{49 + 4 + 64}$$

$$= \sqrt{117}$$

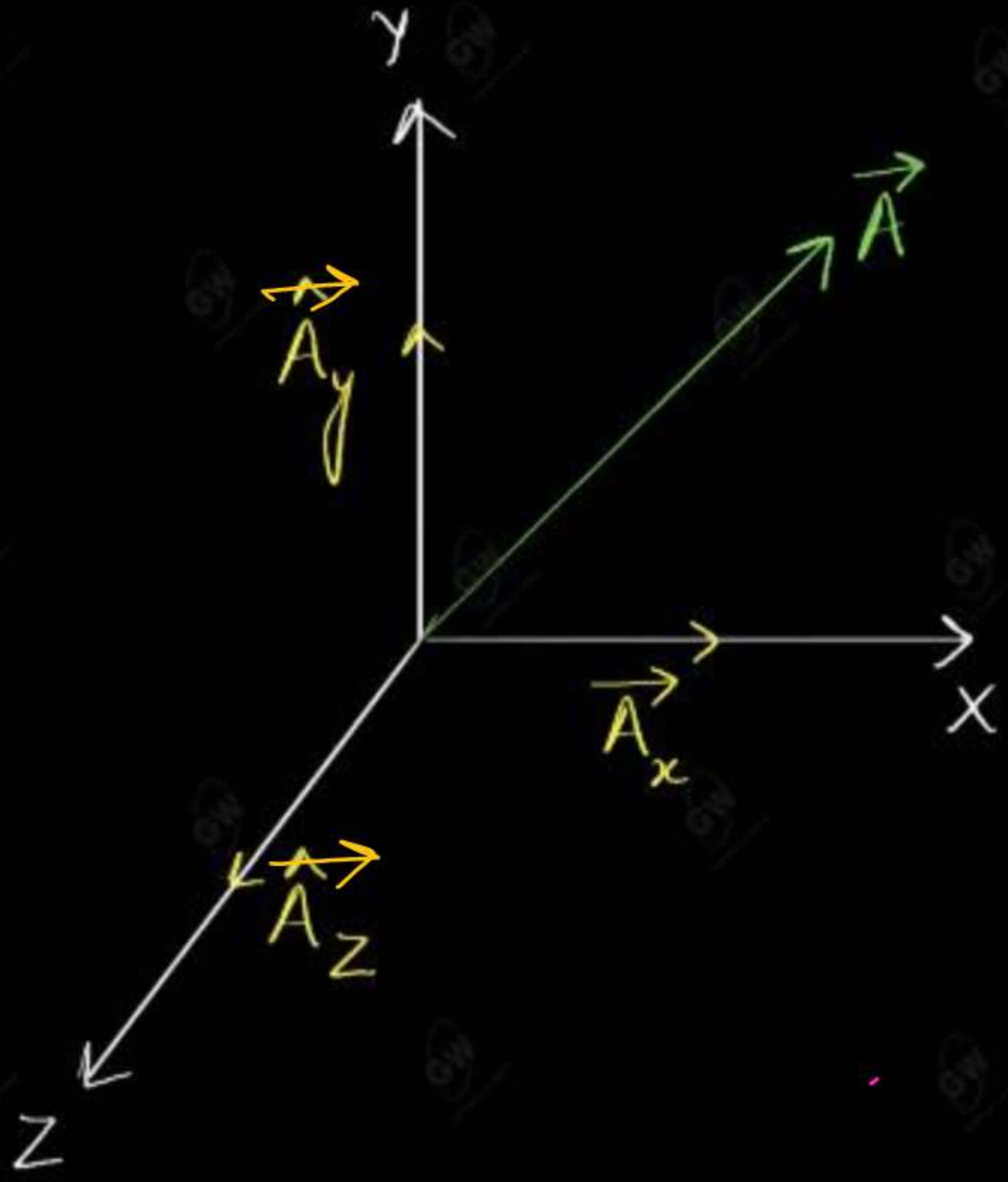
$$\therefore \hat{A} = \frac{7\hat{a}_x + 2\hat{a}_y + 8\hat{a}_z}{\sqrt{117}}$$

$$= \frac{7}{\sqrt{117}}\hat{a}_x + \frac{2}{\sqrt{117}}\hat{a}_y + \frac{8}{\sqrt{117}}\hat{a}_z$$

Ans

# Components of Vector

## In 3D



**Vector = [Vector's magnitude] · Unit Vector**

$$\vec{A} = \vec{A_x} + \vec{A_y} + \vec{A_z}$$

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$\vec{A_x}$ : Component of $\vec{A}$ along X-axis.

$\vec{A_y}$ : Component of $\vec{A}$ along Y-axis.

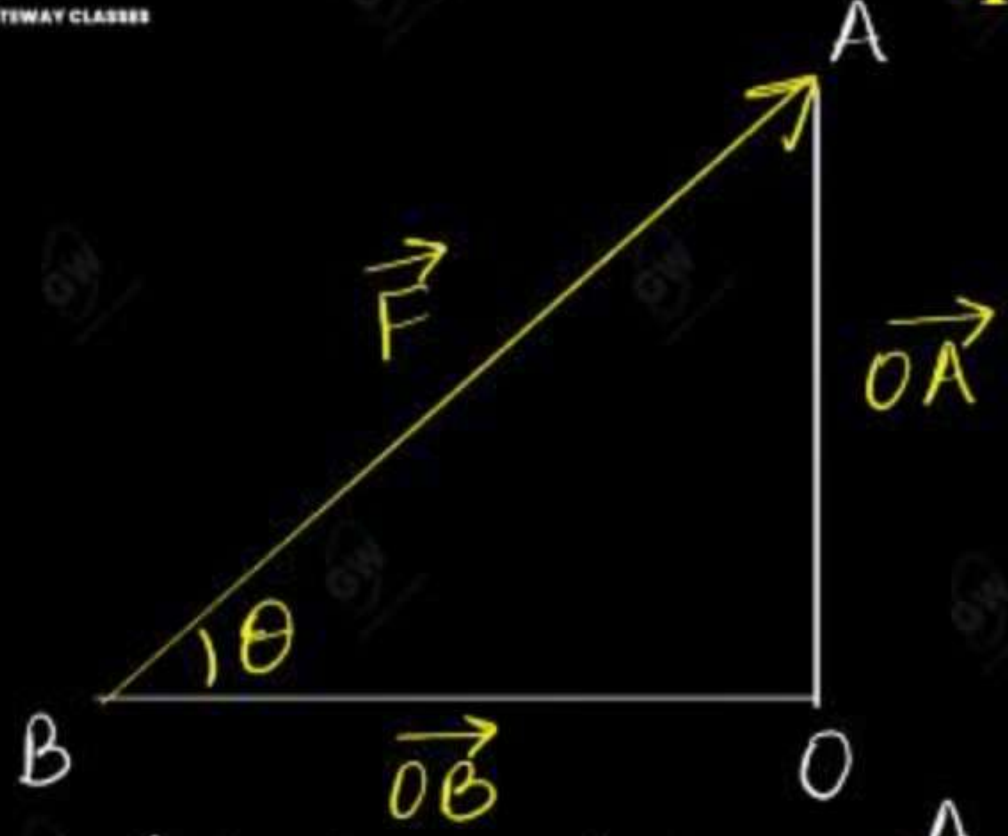
$\vec{A_z}$ : Component of $\vec{A}$ along Z-axis.

$\hat{\imath}$ : Unit vector along X-axis.

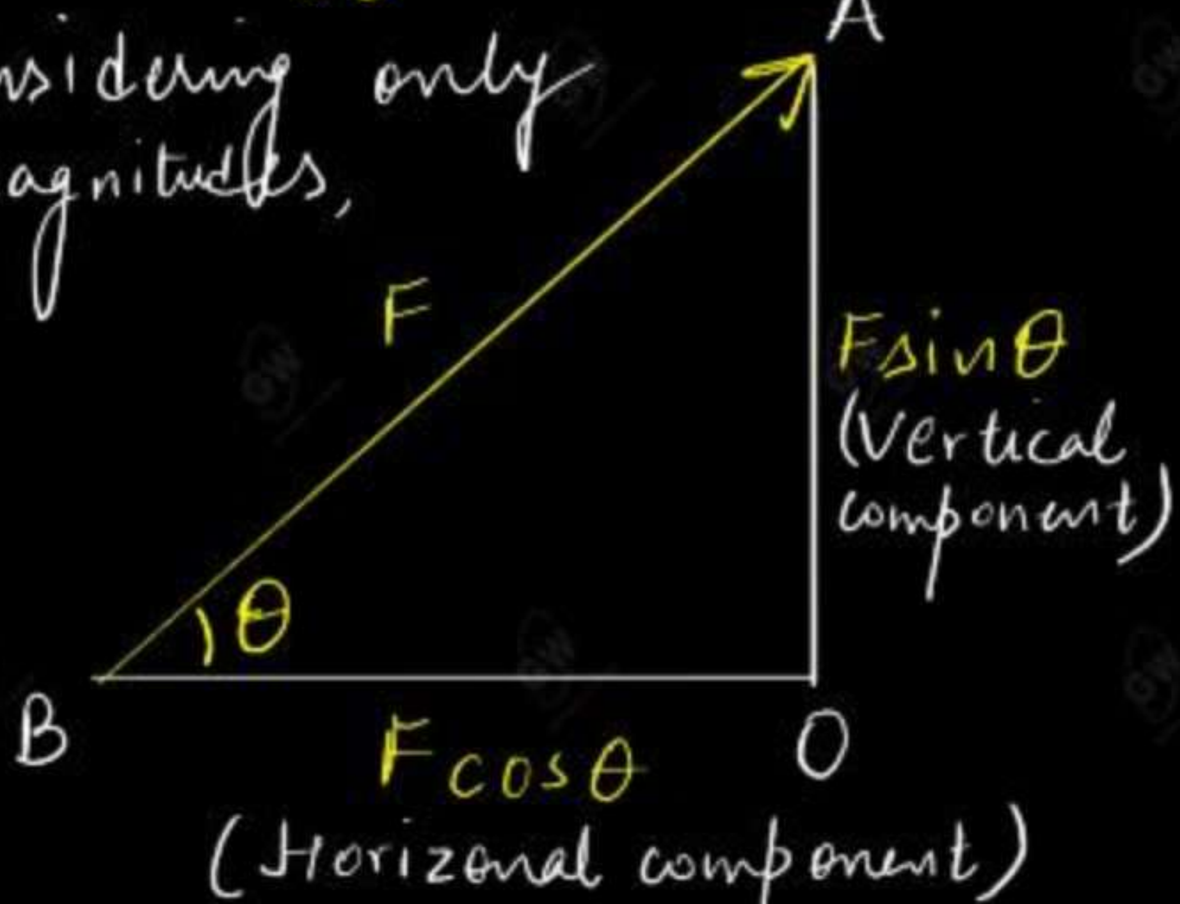
$\hat{\jmath}$ : Unit vector along Y-axis.

$\hat{k}$ : Unit vector along Z-axis.

# Resolution of Vector


A
$\vec{F}$
$\vec{OA}$
$\theta$
B
$\vec{OB}$
O

Considering only magnitudes,


A
F
$F\sin\theta$
(Vertical component)
$\theta$
B
$F\cos\theta$
O
(Horizontal component)

- It is useful to find the components of a vector **F** in two mutually perpendicular directions.
- This process is known as **resolving a vector** into components.

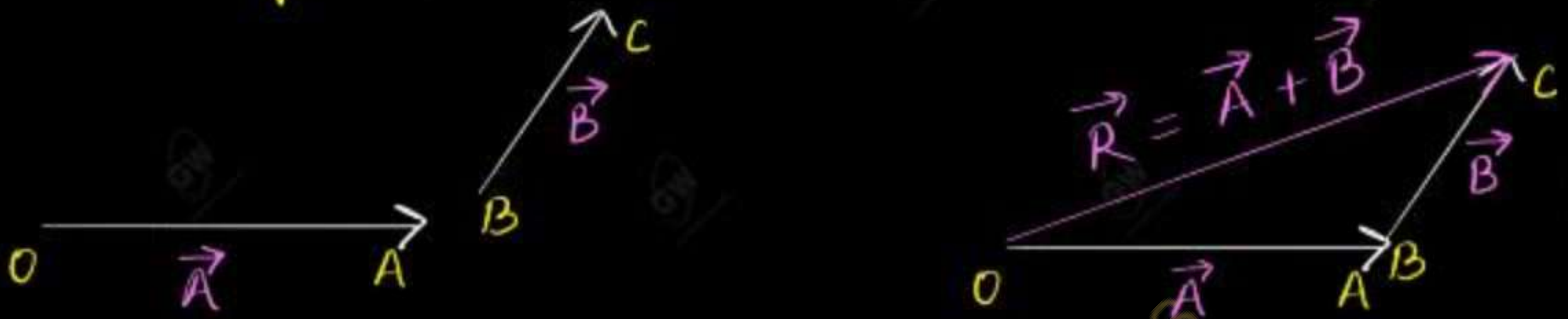
$$\sin\theta = \frac{OA}{BA} = \frac{OA}{F}$$

$$OA = F\sin\theta$$

$$\cos\theta = \frac{OB}{BA} = \frac{OB}{F}$$

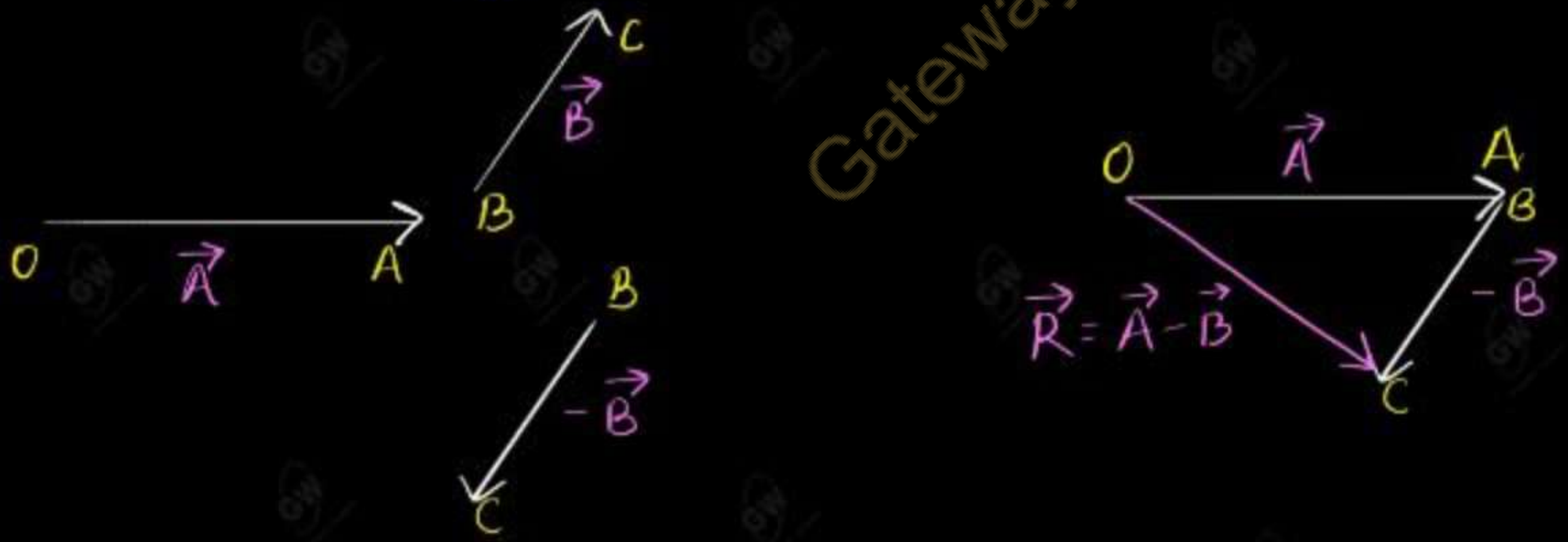
$$OB = F\cos\theta$$

## 1) Addition of Vectors


O
$\vec{A}$
A
B
$\vec{B}$
C
$\vec{R} = \vec{A} + \vec{B}$
O
$\vec{A}$
A
B
$\vec{B}$
C

$$\vec{R} = \vec{A} + \vec{B}$$

## 2) Subtraction of Vectors


O
$\vec{A}$
A
B
$\vec{B}$
C
B
$-\vec{B}$
C
O
$\vec{A}$
A
B
$-\vec{B}$
C
$\vec{R} = \vec{A} - \vec{B}$

$$\vec{R} = \vec{A} + (-\vec{B})$$

or

$$\vec{R} = \vec{A} - \vec{B}$$

## ✓(3) MULTIPLICATION OF VECTORS

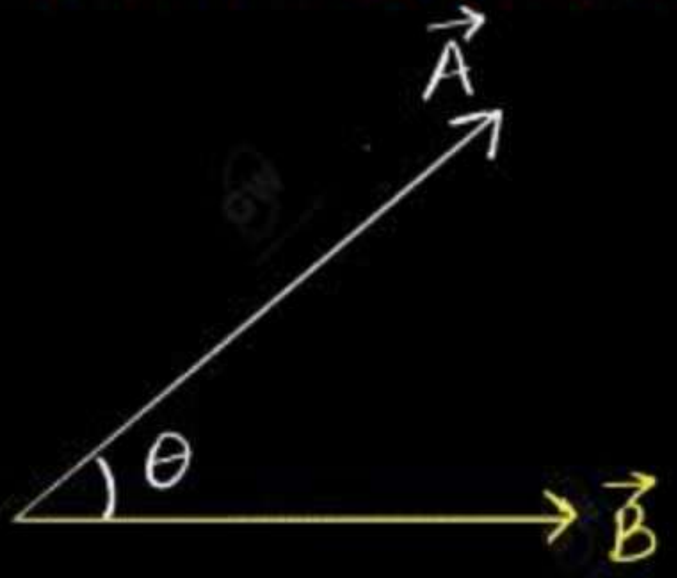
1. SCALAR PRODUCT ( DOT PRODUCT) $\vec{A}\cdot\vec{B}$
2. VECTOR PRODUCT ( CROSS PRODUCT) $\vec{A}\times\vec{B}$
3. SCALAR TRIPLE PRODUCT $\vec{A}\cdot(\vec{B}\times\vec{C})$
4. VECTOR TRIPLE PRODUCT $\vec{A}\times(\vec{B}\times\vec{C})$

## (4) Division of vectors

$\frac{\text{Vector}}{\text{Vector}}$ → Not defined in vector Algebra

$\frac{\text{vector}}{\text{scalar}}$ → is defined

# 1. SCALAR PRODUCT (DOT PRODUCT)



$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

$$= \underbrace{AB\cos\theta}_{\rightarrow\ \text{Scalar Quantity}}$$

## Example

(1) $W = \vec{F}\cdot\vec{s}$

$= Fs\cos\theta$

$W$ = Work done

$F$ = Force

$s$ = displacement

(2) $P = \vec{F}\cdot\overline{v}$

$= Fv\cos\theta$

$P$ = Power

$F$ = Force

$v$ = velocity

## Geometrical meaning of Scalar Product

**Dot product can be regarded as the product of two quantities:** $\rightarrow\ AB\cos\theta = B\,\underline{A\cos\theta}$

a. **The magnitude of one of the vectors**

b. **The scalar component of the second vector along the direction of the first vector**

# Properties of Scalar Product

1. The scalar product is **commutative.**

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\vec{B}\cdot\vec{A} = BA\cos\theta$$

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

2. The scalar product is **distributive** over addition.

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$$

3. The scalar product of two **perpendicular vectors** is **zero**.

$$\vec{A}\cdot\vec{B} = AB\cos 90^\circ$$

$$\vec{A}\cdot\vec{B} = 0$$

## 4.Scalar Product of Unit Vectors

The scalar product of two same unit vectors is one and two different unit vectors is zero.

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = (1)(1)\cos 0^\circ = 1$$

$$\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = (1)(1)\cos 90^\circ = 0$$

# Calculating Scalar Product using Components

**Let**

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

**Then** $\vec{A}\cdot\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$

$$\vec{A}\cdot\vec{B} = A_x\hat{\imath}\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) + A_y\hat{\jmath}\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$= A_xB_x\,\hat{\imath}\cdot\hat{\imath} + A_xB_y\,\hat{\imath}\cdot\hat{\jmath} + A_xB_z\,\hat{\imath}\cdot\hat{k}$$

$$+A_yB_x\,\hat{\jmath}\cdot\hat{\imath} + A_yB_y\,\hat{\jmath}\cdot\hat{\jmath} + A_yB_z\,\hat{\jmath}\cdot\hat{k}$$

$$+A_zB_x\,\hat{k}\cdot\hat{\imath} + A_zB_y\,\hat{k}\cdot\hat{\jmath} + A_zB_z\,\hat{k}\cdot\hat{k}$$

$$= A_xB_x(1) + A_xB_y(0) + A_xB_z(0)$$

$$+A_yB_x(0) + A_yB_y(1) + A_yB_z(0)$$

$$+A_zB_x(0) + A_zB_y(0) + A_zB_z(1)$$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

**Que-3) If A = $\alpha a_x + 2a_y + 10\,a_z$ and $B = 4\alpha a_x + 8a_y - 2\alpha\, a_z$, find the value of $\alpha$ for which the two vectors become perpendicular.**

For $\vec{A} \perp \vec{B}$, $\theta = 90^\circ$

$\therefore \vec{A}\cdot\vec{B} = AB\cos 90^\circ = 0$

$\because \cos 90^\circ = 0$

$\therefore \vec{A}\cdot\vec{B} = 0$

$$(\alpha\hat{a}_x + 2\hat{a}_y + 10\hat{a}_z)\cdot(4\alpha\hat{a}_x + 8\hat{a}_y - 2\alpha\hat{a}_z) = 0$$

$$4\alpha^2 + 16 - 20\alpha = 0$$

$\because \hat{a}_x\cdot\hat{a}_x = \hat{a}_y\cdot\hat{a}_y = \hat{a}_z\cdot\hat{a}_z = 1$

$\hat{a}_x\cdot\hat{a}_y = \hat{a}_x\cdot\hat{a}_z = \hat{a}_y\cdot\hat{a}_z = 0$

$\therefore$

$$4\alpha^2 - 20\alpha + 16 = 0$$

$$4(\alpha^2 - 5\alpha + 4) = 0$$

$$\alpha = 1 \text{ or } \alpha = 4$$

Ans

# PROJECTION OF A VECTOR ON ANOTHER VECTOR
## ( Application of Dot Product )

Projection of $\vec{A}$ on $\vec{B}$ $= \dfrac{(\vec{A}\cdot\vec{B})}{|\vec{B}|^2}\cdot\vec{B}$

Projection of $\vec{B}$ on $\vec{A}$ $= \dfrac{(\vec{A}\cdot\vec{B})}{|\vec{A}|^2}\cdot\vec{A}$

Que-4) Given two vectors

$\vec{A} = 4\hat{j} + 10\hat{k}$ and

$\vec{B} = 2\hat{i} + 3\hat{j}$

Find the projection of $\vec{A}$ on $\vec{B}$.

$$\boxed{\text{Projection of } \vec{A} \text{ on } \vec{B} = \frac{\vec{A}\cdot\vec{B}}{|\vec{B}|}}$$

Now, $\vec{A}\cdot\vec{B} = (4\hat{j} + 10\hat{k})\cdot(2\hat{i} + 3\hat{j})$

$= 12\,\hat{j}\cdot\hat{j}$

$= 12$

$|\vec{B}| = \sqrt{(2)^2 + (3)^2}$

$= \sqrt{13}$

Proj. of $\vec{A}$ on $\vec{B}$ $= \dfrac{12}{\sqrt{13}}$ Ans.

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**Given two vectors**

$$\vec{P} = (0, 6, -9) \text{and } \vec{Q} = (-2, 0, 3)$$

**Find the projection of $\vec{P}$ on $\vec{Q}$.**

$$\boxed{\text{Projection of } \vec{P} \text{ on } \vec{Q} = \frac{\left(\vec{P}\cdot\vec{Q}\right)}{|\vec{Q}|^2}\cdot\vec{Q}} \quad - \text{①}$$

$$\vec{P} = 0\hat{a}_x + 6\hat{a}_y - 9\hat{a}_z$$

$$\vec{Q} = -2\hat{a}_x + 0\hat{a}_y + 3\hat{a}_z$$

$$\vec{P}\cdot\vec{Q} = \left(0\hat{a}_x + 6\hat{a}_y - 9\hat{a}_z\right)\left(-2\hat{a}_x + 0\hat{a}_y + 3a_z\right) = -27$$

$$|\vec{Q}|^2 = \left(\sqrt{(-2)^2 + (0)^2 + (3)^2}\right)^2 = 13$$

Put in ①,

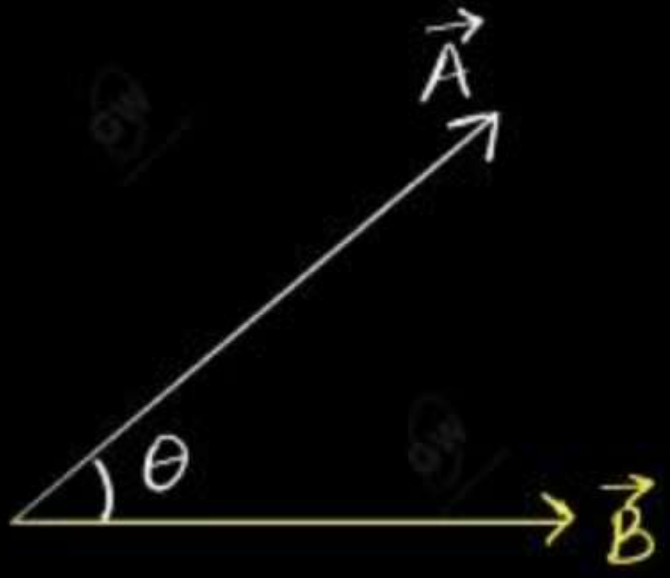
$$\boxed{\text{Projection of } \vec{P} \text{ on } \vec{Q} = \frac{\left(\vec{P}\cdot\vec{Q}\right)}{|\vec{Q}|^2}\cdot\vec{Q}}$$

$$= \frac{-27}{13}\left(-2\hat{a}_x + 0\,\hat{a}_y + 3a_z\right)$$

$$= \frac{27}{13}\left(2\hat{a}_x - 3\hat{a}_z\right)$$

$$= \frac{54}{13}\hat{a}_x - \frac{81}{13}\hat{a}_z$$ Ans

# 2. VECTOR PRODUCT ( CROSS PRODUCT)

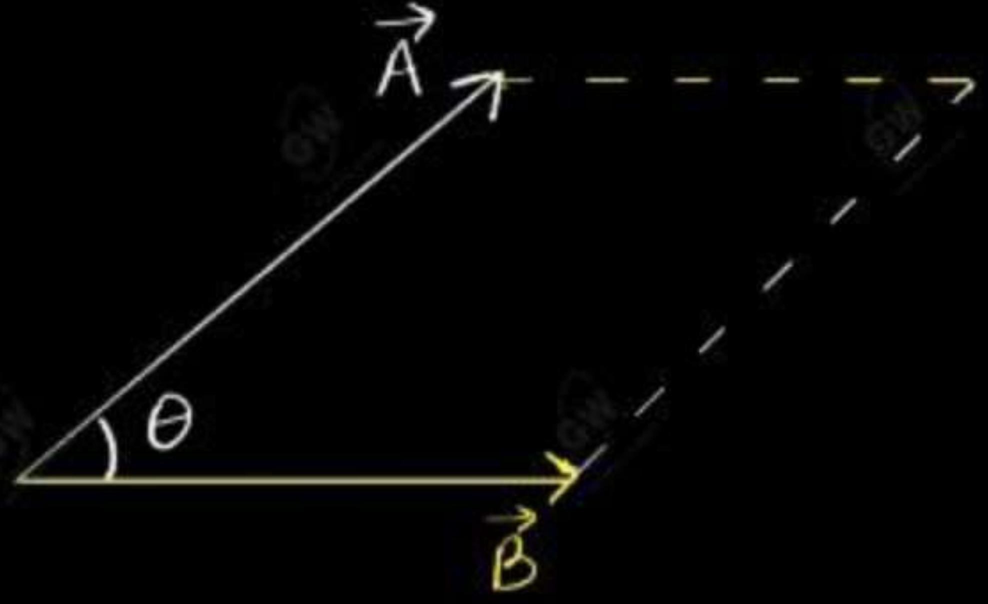

$\vec{A}$
$\theta$
$\vec{B}$

$$\underline{\vec{A} \times \vec{B}} = |\vec{A}||\vec{B}| \sin\theta \cdot \hat{n}$$

where $\hat{n}$ is the unit vector to the plane containing $\vec{A}$ and $\vec{B}$

Vector Quantity

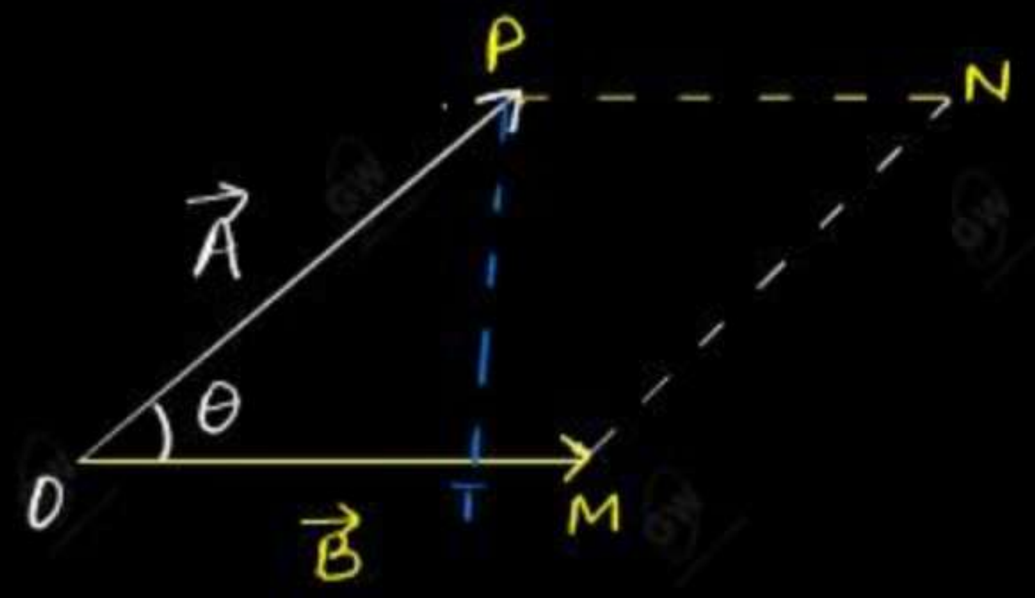
## Geometrical meaning of Vector Product


$\vec{A}$
$\theta$
$\vec{B}$

$|\vec{A} \times \vec{B}|$ = Area of parallelogram made by two vectors

# Geometrical meaning of Vector Product

$|\vec{A}\times\vec{B}|$ = Area of parallelogram made by two vectors


P
N
$\vec{A}$
$\theta$
O
$\vec{B}$
T
M

$= \text{Base} \times \text{Height}$

$= (OM)(PT)$

$= B(A\sin\theta)$

$= AB\sin\theta$

$= |\vec{A}\times\vec{B}|$

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**Find the value of**

$$(3\hat{a}_x + 6\hat{a}_y) \times (2\hat{a}_x + 3\hat{a}_y + 5\hat{a}_z),$$

**where × denotes the cross product.**

$$\vec{A} = 3\hat{a}_x + 6\hat{a}_y$$

$$\vec{B} = 2\hat{a}_x + 3\hat{a}_y + 5\hat{a}_z$$

$$\therefore \quad \vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 3 & 6 & 0 \\ 2 & 3 & 5 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = [(6\times 5) - (3\times 0)]\,\hat{a}_x - [(3\times 5) - (2\times 0)]\,\hat{a}_y + [(3\times 3) - (2\times 6)]\,\hat{a}_z$$

$$= 30\hat{a}_x - 15\hat{a}_y - 3\hat{a}_z$$

Ans

# Properties of Vector Product

1. The vector product is anti-commutative.

$$\vec{A} \times \vec{B} = AB \sin\theta \;\hat{n}$$

$$\vec{B} \times \vec{A} = BA \sin\theta \;(-\hat{n}) = -AB \sin\theta \;\hat{n}$$

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

2. The vector product is distributive over addition.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

3. The magnitude of the vector product of two perpendicular vectors is maximum.

$$|\vec{A} \times \vec{B}| = AB \sin 90^\circ$$

$$|\vec{A} \times \vec{B}| = AB$$

4. The vector product of two parallel vectors is a null vector.

$$\vec{A} \times \vec{B} = AB \sin 0^\circ \;\hat{n}$$

$$\vec{A} \times \vec{B} = \vec{0}$$

5. The vector product of a vector with itself is a null vector.

$$\vec{A} \times \vec{A} = AA \sin 0^\circ \;\hat{n}$$

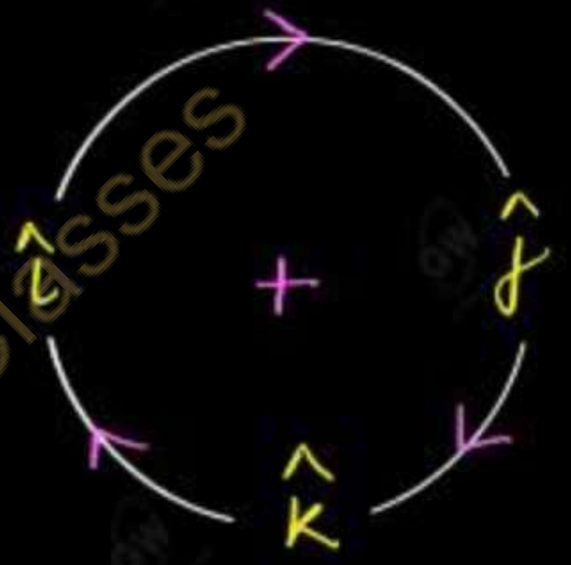
$$\vec{A} \times \vec{A} = \vec{0}$$

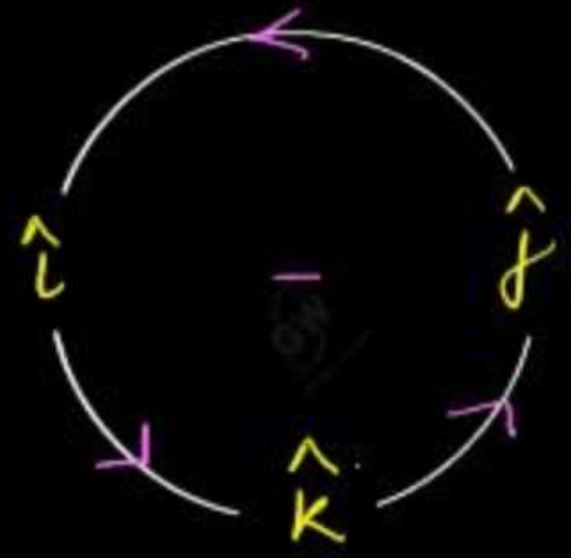
## 6. The vector product of two **same unit vectors** is a null vector.

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k}$$

$$= (1)(1)\sin 0^\circ\ \hat{n} = \vec{0}$$

## 7. The vector product of **two different unit vectors** is a third unit vector.





$\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}$

$\hat{k} \times \hat{j} = -\hat{i}$

$\hat{i} \times \hat{k} = -\hat{j}$

# Calculating Vector Product using Components

**Let**

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

**Then** $\vec{A} \times \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$

$$\vec{A} \times \vec{B}$$
$$= A_x\hat{\imath} \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) + A_y\hat{\jmath} \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$
$$+ A_z\hat{k} \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$
$$= A_xB_x\,\underline{\hat{\imath}\times\hat{\imath}} + A_xB_y\,\underline{\hat{\imath}\times\hat{\jmath}} + A_xB_z\,\underline{\hat{\imath}\times\hat{k}} + A_yB_x\,\underline{\hat{\jmath}\times\hat{\imath}} + A_yB_y\,\underline{\hat{\jmath}\times\hat{\jmath}}$$
$$+ A_yB_z\,\underline{\hat{\jmath}\times\hat{k}} + A_zB_x\,\underline{\hat{k}\times\hat{\imath}} + A_zB_y\,\underline{\hat{k}\times\hat{\jmath}} + A_zB_z\,\underline{\hat{k}\times\hat{k}}$$

$$= A_xB_x(0) + A_xB_y(\hat{k}) + A_xB_z(-\hat{\jmath}) + A_yB_x(-\hat{k}) + A_yB_y(0)$$
$$+ A_yB_z(\hat{\imath}) + A_zB_x(\hat{\jmath}) + A_zB_y(-\hat{\imath}) + A_zB_z(0)$$

$$= (A_yB_z - A_zB_y)(\hat{\imath}) + (A_zB_x - A_xB_z)(\hat{\jmath})$$
$$+ (A_xB_y - A_yB_x)(\hat{k})$$

$$= (A_yB_z - A_zB_y)(\hat{\imath}) - (A_xB_z - A_zB_x)(\hat{\jmath})$$
$$+ (A_xB_y - A_yB_x)(\hat{k})$$

$$\boxed{\underline{\vec{A} \times \vec{B}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}}$$

→ Vector Quantity

∴ ↳ $\hat{\imath}, \hat{\jmath}, \hat{k}$ in Ist row.

# 3. SCALAR TRIPLE PRODUCT

$$\vec{A} \cdot (\vec{B} \times \vec{C})$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\vec{C} = C_x \hat{i} + C_y \hat{j} + C_z \hat{k}$$

Now,

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\vec{B} \times \vec{C} = (B_y C_z - B_z C_y)\hat{i} - (B_x C_z - B_z C_x) + (B_x C_y - B_y C_x)\hat{k}$$

Now,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = A_x (B_y C_z - B_z C_y) - A_y (B_x C_z - B_z C_x) + A_z (B_x C_y - B_y C_x).$$

$$\boxed{\underbrace{\vec{A} \cdot (\vec{B} \times \vec{C})}_{\text{Scalar Quantity}} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}}$$

$\therefore$ no $\hat{i}, \hat{j}, \hat{k}$ in $I^{st}$ row.

# EXAMPLE - SCALAR TRIPLE PRODUCT

Que-5) Find $\vec{A}\cdot(\vec{B}\times\vec{C}) = ?$

$$\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$$
$$\vec{B} = \hat{i} - 2\hat{j} + 4\hat{k}$$
$$\vec{C} = -\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= 2\left[(-2\times2)-(1\times4)\right] - 3\left[(1\times2)-(-1\times4)\right] +$$
$$-1\left[(1\times1)-(-1\times-2)\right]$$

$$= 2\left[-4-4\right] - 3\left[2+4\right] - 1\left[1-2\right]$$

$$= 2(-8) - 3(6) - 1(-1)$$

$$= -16 - 18 + 1$$

$$= -33$$ Ans

# 4. VECTOR TRIPLE PRODUCT

$$\vec{A} \times (\vec{B} \times \vec{C})$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$
$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$
$$\vec{C} = C_x \hat{i} + C_y \hat{j} + C_z \hat{k}$$

Then

$$\underbrace{\vec{A}\times(\vec{B}\times\vec{C})}_{\text{Vector Quantity}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ (B_y C_z - B_z C_y) & \overset{\text{Note}}{\boxed{(B_z C_x - B_x C_z)}} & (B_x C_y - B_y C_x) \end{vmatrix}$$

$\therefore$ $\hat{i}, \hat{j}, \hat{k}$ in $I^{st}$ row.

# EXAMPLE-VECTOR TRIPLE PRODUCT

Que-6) $\vec{A}\times(\vec{B}\times\vec{C}) = ?$

$\vec{A} = 2\hat{i} - 3\hat{j}$

$\vec{B} = \hat{i} + \hat{j} - \hat{k}$

$\vec{C} = 3\hat{i} - \hat{k}$

$$\underline{\vec{A}\times(\vec{B}\times\vec{C})} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ (B_yC_z - B_zC_y) & (B_zC_x - B_xC_z) & (B_xC_y - B_yC_x) \end{vmatrix}$$

$$\underline{\vec{A}\times(\vec{B}\times\vec{C})} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 0 \\ (B_yC_z - B_zC_y) & (B_zC_x - B_xC_z) & (B_xC_y - B_yC_x) \end{vmatrix}$$

$$\begin{array}{c|ccc} & x & y & z \\ \hline B & 1 & 1 & -1 \\ C & 3 & 0 & -1 \end{array}$$

$$B_yC_z - B_zC_y = 1(-1) - (-1)(0)$$
$$= -1$$

$$B_zC_x - B_xC_z = -1(3) - 1(-1)$$
$$= -3+1$$
$$= -2$$

$$B_xC_y - B_yC_x = 1(0) - 1(3)$$
$$= -3$$

$$\underline{\vec{A}\times(\vec{B}\times\vec{C})} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 0 \\ -1 & -2 & -3 \end{vmatrix}$$

$$= \hat{i}\Big[(-3\times-3)-(-2\times0)\Big]-\hat{j}\Big[(2\times-3)-(-1\times0)\Big]+\hat{k}\Big[(2\times-2)-(-1\times-3)\Big]$$

$$= \hat{i}\,[9-0]-\hat{j}\,[-6-0]+\hat{k}\,[-4-3]$$

$$= 9\hat{i}+6\hat{j}-7\hat{k}$$ Ans.

# AKTU B.Tech Sem-3

## EE & EE ALLIED

# EMFT

## ( ELECTROMAGNETIC FIELD THEORY )

### Unit-1 : Lec-2

**Topics to be covered**

- Coordinate System
  - Transformation:
    1. Point to Point Transformation
    2. Vector to Vector Transformation
- PYQs

GATEWAY CLASSES

**By Swati Ma'am**

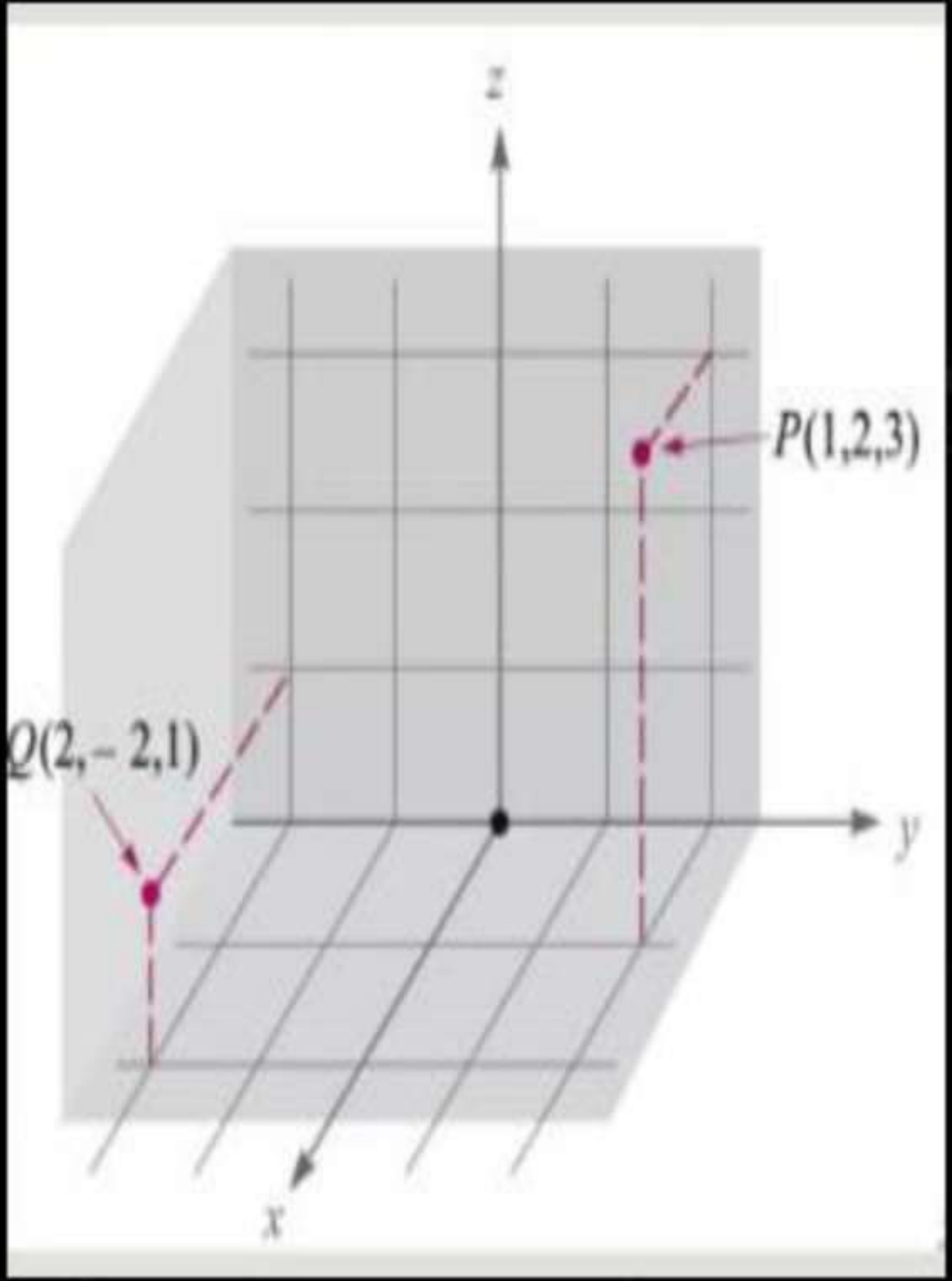
- M.Tech (Honours)
- B.Tech (Honours)
- GATE Qualified

# Coordinate Systems and Transformation

- In order to describe a point or a vector accurately in terms of lengths , angles and projections etc ,there are Three types of Co-ordinate systems as different problems have different shapes.

- ➢ Rectangular (or) Cartesian Co-ordinate System (Orbits ) $P(x, y, z)$
- ➢ Cylindrical Co-ordinate System (Buildings,Tunnels ,Pipes ,Towers ) $P(\rho, \underbrace{\phi}_{\text{One angle}}, z)$
- ➢ Spherical Co-ordinate System ( Stars ,Sound waves ) $P(r, \underbrace{\theta, \phi}_{\text{Two angles}})$

# 1. Rectangular (or) Cartesian Co-ordinate System



$P(x, y, z)$

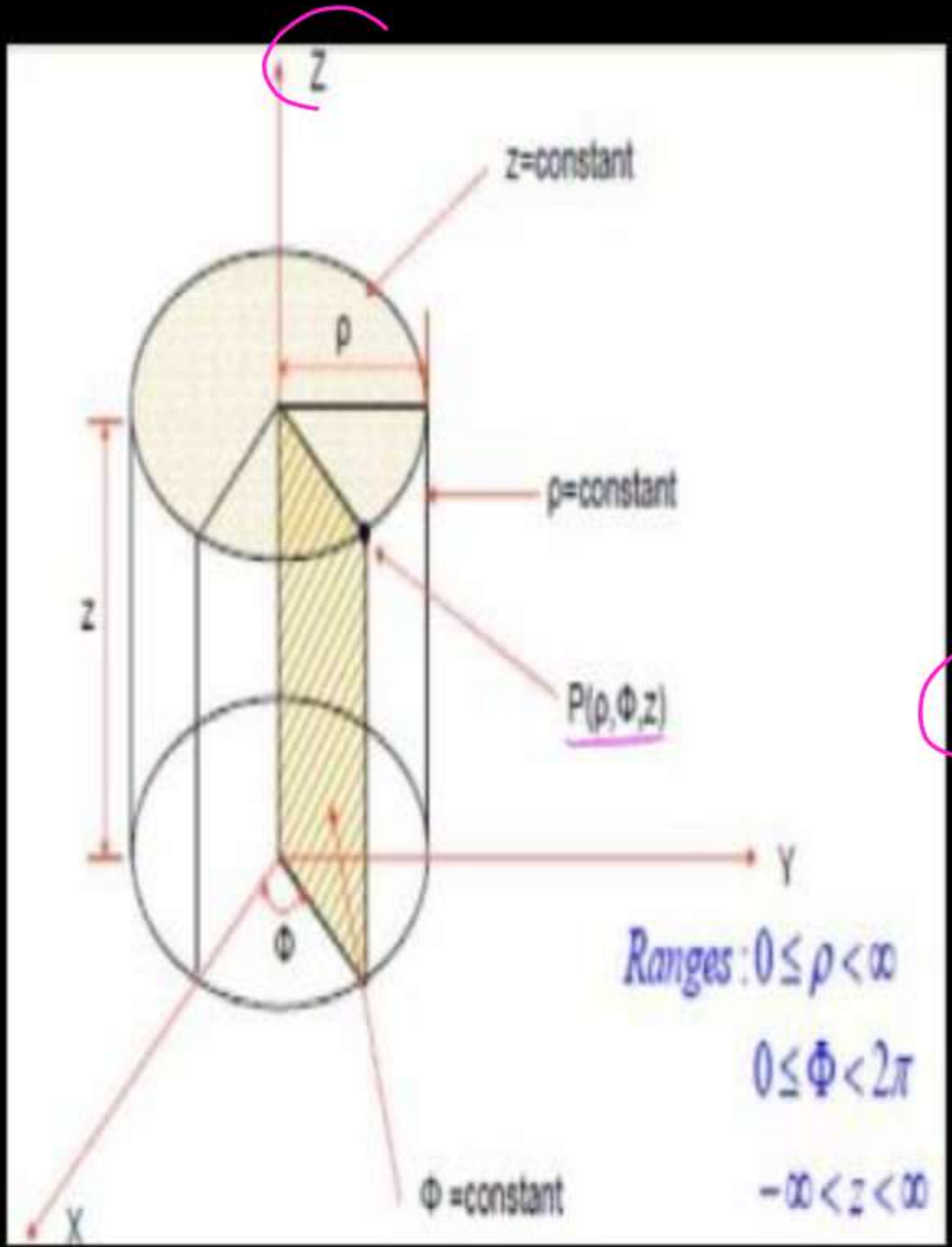
$x$ :- Distance of P from X axis

$y$ :- Distance of P from Y axis

$Z$ :- Distance of P from Z axis

| | |
|---|---|
| Position of point (P) | $x, y, z$ |
| Unit Vectors | $\hat{a}_x, \hat{a}_y, \hat{a}_z$ |
| Position Vector | $\vec{r} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ |

# 2. Cylindrical Co-ordinate System


Z
z=constant
ρ
ρ=constant
z
P(ρ,Φ,z)
Y
Φ
Ranges : $0 \le \rho < \infty$
$0 \le \Phi < 2\pi$
$-\infty < z < \infty$
Φ =constant
X

$P(\rho, \phi, z)$

$\rho$ :- Radius of an imaginary cylinder passing through P

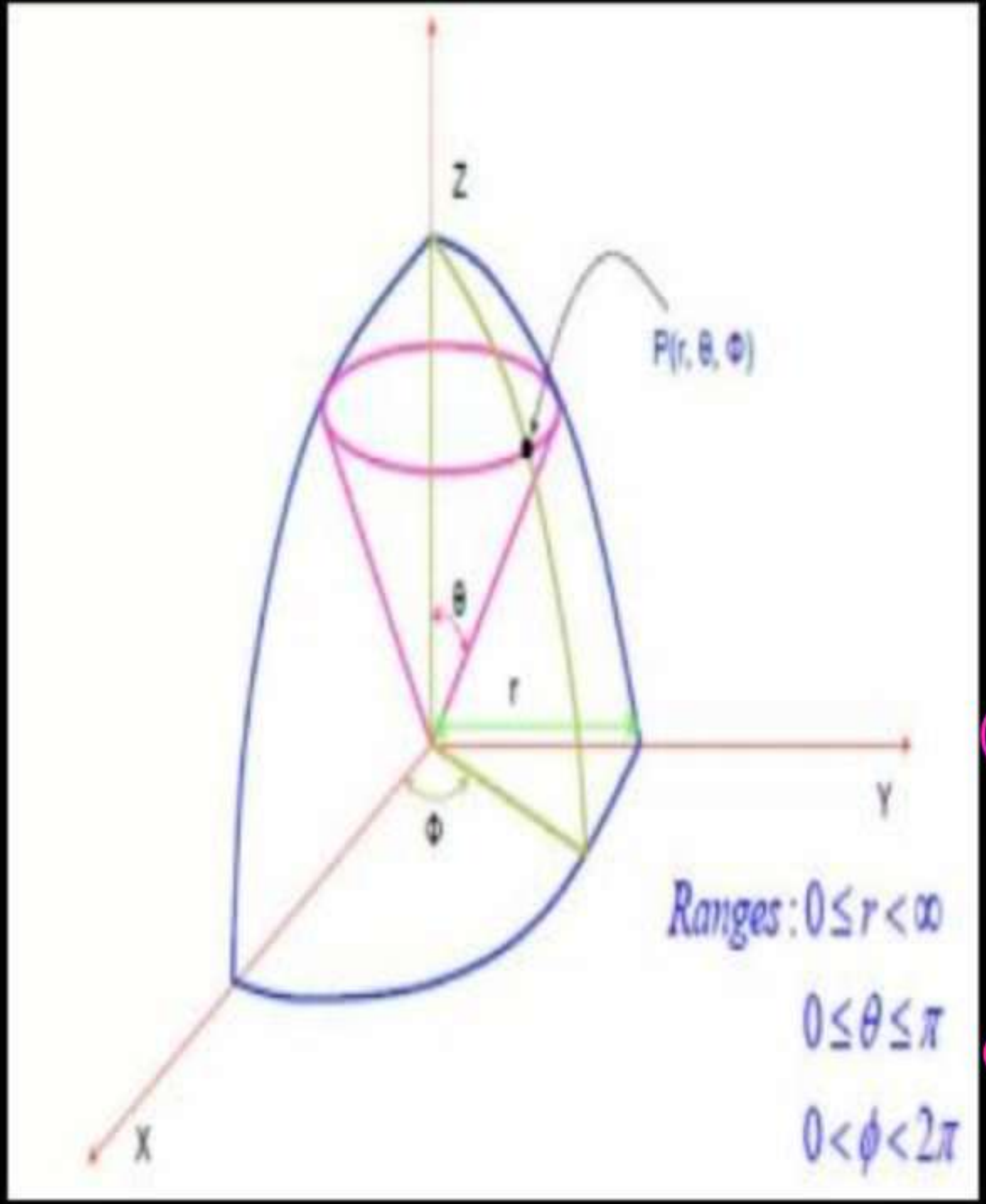
$\phi$ :- Angle between X axis and a vertical intersecting plane passing through P.

0 to $2\pi$

$z$ :- Vertical distance of P from XY plane.

| | |
|---|---|
| Position of point (P) | $\rho, \phi, Z$ |
| Unit Vectors | $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ |
| Position Vector | $\vec{r} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$ |

# 3.Spherical Co-ordinate System



Ranges: $0 \le r < \infty$

$0 \le \theta \le \pi$

$0 < \phi < 2\pi$

$P(r, \theta, \phi)$

r :- Radius of imaginary sphere passing through point P.

$\theta$ :- Half angle of cone passing through P.

0 to $\pi$

$\phi$ :- Angle between X axis and a vertical intersecting plane passing through P.

0 to $2\pi$

| | |
|---|---|
| Position of point (P) | $r, \theta, \phi$ |
| Unit Vectors | $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ |
| Position Vector | $\vec{r} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$ |

# Transformations

Point to Point

Vector to vector

## (I) Point to Point Transformation

### 1) Cartesian → Cylindrical

$(x, y, z) \longrightarrow (\rho, \phi, z)$

Learn

$$\boxed{\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned}}$$

### 2) Cylindrical → Cartesian

$(\rho, \phi, z) \longrightarrow (x, y, z)$

Learn

$$\boxed{\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}}$$

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## 3) Cartesian → Spherical

$$(x, y, z) = (r, \theta, \phi)$$

Learn

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}}$$

## 4) Spherical → Cartesian

$$(r, \theta, \phi) = (x, y, z)$$

Learn

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$$

## 5) Cylindrical → Spherical

$$(\rho, \phi, z) \longrightarrow (r, \theta, \phi)$$

Learn

$$\boxed{\begin{aligned} r &= \sqrt{\rho^2 + z^2} \\ \theta &= \tan^{-1}\left(\frac{\rho}{z}\right) \\ \phi &= \phi \end{aligned}}$$

## 6) Spherical → Cylindrical

$$(r, \theta, \phi) \longrightarrow (\rho, \phi, z)$$

Learn

$$\boxed{\begin{aligned} \rho &= r \sin\theta \\ \phi &= \phi \\ z &= r\cos\theta \end{aligned}}$$

# AKTU PYQ-2020-21

**Que-1) Transform the point $P(-2,3,4)$ into cylindrical coordinate system**

$P(-2,3,4)$ is in cartesian co-ordinate system.

converting it to cylindrical $(\rho, \phi, z)$,

$$\rho = \sqrt{x^2+y^2} = \sqrt{(-2)^2+(3)^2} = \sqrt{13} = 3.60$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{-2}\right) = -56.30^\circ$$

$$z = z = 4$$

$$\therefore \quad P(3.60, -56.30^\circ, 4) \qquad \text{Ans}$$

# AKTU PYQ-2024-25

**Que-2) Given point $P(-2, 6, 3)$ express $P$ in cylindrical and spherical coordinates.**

$P(-2,6,3)$ is in cartesian co-ordinate system.

converting it to cylindrical $(\rho, \phi, z)$,

$$\rho = \sqrt{x^2+y^2} = \sqrt{(-2)^2+(6)^2} = \sqrt{40} = 2\sqrt{10}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{-2}\right) = -71.56^\circ$$

$$z = z = 3$$

$$\therefore \quad P\left(2\sqrt{10}, -71.56^\circ, 3\right)$$ Ans

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$P(-2,6,3)$ is in cartesian co-ordinate system.

converting it to spherical $(r, \theta, \phi)$,

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{(-2)^2+(6)^2+(3)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{3}{7}\right) = 64.62^\circ$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{-2}\right) = -71.56^\circ$$

$$\therefore P(7, 64.62^\circ, -71.56^\circ) \text{ Ans}$$

AKTU PYQ-2024-25

**Que-3) Convert the cylindrical coordinates (3, 20°,4) into equivalent spherical coordinates.**

$P(3, 20^\circ, 4)$ is in Cylindrical co-ordinate system.

converting it to spherical $(r, \theta, \phi)$,

$$r = \sqrt{\rho^2 + z^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right) = \tan^{-1}\left(\frac{3}{4}\right) = 36.86^\circ$$

$$\phi = \phi = 20^\circ$$

$$\therefore \quad P(5, 36.86^\circ, 20^\circ).$$

# AKTU PYQ-2022-23

**Que-4) Find the distance between the pair of points:**

$$\left(4, \frac{\pi}{3}, 5\right) \text{ and } \left(-1, \frac{\pi}{2}, 3\right)$$

First convert these points from cylindrical to cartesian co-ordinate system, then apply distance formula.

(i) $\left(4, \frac{\pi}{3}, 5\right)$

$$x_1 = \rho\cos\phi = 4\cos\frac{\pi}{3} = 4 \times \frac{1}{2} = 2$$

$$y_1 = \rho\sin\phi = 4\sin\frac{\pi}{3} = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$z_1 = z = 5$$

$$\therefore (x_1, y_1, z_1) = (2, 2\sqrt{3}, 5)$$

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(ii) $(-1, \frac{\pi}{2}, 3)$

$$x_2 = \rho\cos\phi = -1\cos\frac{\pi}{2} = -1\times 0 = 0$$

$$y_2 = \rho\sin\phi = -1\sin\frac{\pi}{2} = -1\times 1 = -1$$

$$z_2 = z = 3$$

$$\therefore (x_2, y_2, z_2) = (0, -1, 3)$$

$(\underset{x_1}{2}, \underset{y_1}{2\sqrt{3}}, \underset{z_1}{5})$ $(\underset{x_2}{0}, \underset{y_2}{-1}, \underset{z_2}{3})$

Distance formula $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$$= \sqrt{(0-2)^2+(-1-2\sqrt{3})^2+(3-5)^2}$$

$$= \sqrt{4+7.46+4}$$

$$= \sqrt{15.46}$$

$= 3.93$ units Ans

# II) Vector to Vector Transformation

uses → Tables and then → Point to Point Transformation Formulae

## Dot Products of Unit Vectors in Rectangular and Cylindrical Co-ordinate System

Learn

| | $\hat{a}_\rho$ | $\hat{a}_\phi$ | $\hat{a}_z$ |
|---|---|---|---|
| $\hat{a}_x$ | $\cos\phi$ | $-\sin\phi$ | $0$ |
| $\hat{a}_y$ | $\sin\phi$ | $\cos\phi$ | $0$ |
| $\hat{a}_z$ | $0$ | $0$ | $1$ |

## Dot Products of Unit Vectors in Rectangular and Spherical Co-ordinate System

Learn

| | $\hat{a}_r$ | $\hat{a}_\theta$ | $\hat{a}_\phi$ |
|---|---|---|---|
| $\hat{a}_x$ | $\sin\theta\cos\phi$ | $\cos\theta\cos\phi$ | $-\sin\phi$ |
| $\hat{a}_y$ | $\sin\theta\sin\phi$ | $\cos\theta\sin\phi$ | $\cos\phi$ |
| $\hat{a}_z$ | $\cos\theta$ | $-\sin\theta$ | $0$ |

Que-5) Express the vector field $\vec{W} = (x - y)\,\hat{a}_y$ in cylindrical coordinates.

vector transformation Cartesian $\longrightarrow$ cylindrical

$(W_x, W_y, W_z)$ $\qquad$ $(W_\rho, W_\phi, W_z)$

$$\vec{W} = W_x\hat{a}_x + W_y\hat{a}_y + W_z\hat{a}_z \longrightarrow \vec{W} = W_\rho\hat{a}_\rho + W_\phi\hat{a}_\phi + W_z\hat{a}_z$$

$$\vec{W} = (x-y)\,\hat{a}_y$$

$$W_\rho = \vec{W}\,\hat{a}_\rho = (x-y)\overbrace{\hat{a}_y \cdot \hat{a}_\rho}^{\text{Use Table}} = (x-y)\sin\phi$$

$$W_\phi = \vec{W}\,\hat{a}_\phi = (x-y)\overbrace{\hat{a}_y\,\hat{a}_\phi}^{} = (x-y)\cos\phi$$

$$W_z = \vec{W}\,\hat{a}_z = (x-y)\overbrace{\hat{a}_y\,\hat{a}_z}^{} = (x-y)\cdot 0 \quad = 0$$

$$\vec{W} = W_\rho \cdot \hat{a}_\rho + W_\phi \cdot \hat{a}_\phi + W_z \hat{a}_z$$

$$= (x-y)\sin\phi \cdot \hat{a}_\rho + (x-y)\cos\phi \cdot \hat{a}_\phi$$

($x, y$ still are there)

$\vec{W}$ is still in Cartesian coordinates,

We know, Use Point Transformation

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$$\therefore \vec{W} = (\rho\cos\phi - \rho\sin\phi)\sin\phi \cdot \hat{a}_\rho + (\rho\cos\phi - \rho\sin\phi)\cos\phi \cdot \hat{a}_\phi$$

Ans

# AKTU PYQ-2023-24

**Que-6) Organize the vector $\vec{A} = r\sin\theta\ \hat{a}_r$ into Cartesian coordinate system considering $\vec{A}$ is in spherical coordinate system.**

vector transformation spherical ⟶ Cartesian.

$(A_r, A_\theta, A_\phi)$ $(A_x, A_y, A_z)$

$\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\cdot\hat{a}_\phi \longrightarrow \vec{A} = A_x\hat{a}_x + A_y\cdot\hat{a}_y + A_z\cdot\hat{a}_z$

$\vec{A} = r\sin\theta\cdot\hat{a}_r$

Use Table

$A_x = \vec{A}\cdot\hat{a}_x = r\sin\theta\ \underline{\hat{a}_r\cdot\hat{a}_x} = r\sin\theta\ \sin\theta\cos\phi$

$A_y = \vec{A}\cdot\hat{a}_y = r\sin\theta\ \underline{\hat{a}_r\cdot\hat{a}_y} = r\sin\theta\ \sin\theta\sin\phi$

$A_z = \vec{A}\cdot\hat{a}_z = r\sin\theta\ \underline{\hat{a}_r\cdot\hat{a}_z} = r\sin\theta\ \cos\theta$

$$\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$$

$$\vec{A} = r\sin\theta\sin\theta\cos\phi \cdot \hat{a}_x + r\sin\theta\sin\theta\sin\phi \cdot \hat{a}_y + r\sin\theta\cos\theta\,\hat{a}_z .$$

r, θ, φ still are there

But still it is in spherical

∴ we can rewrite, Use Point Transformation

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\vec{A} = x\sin\theta\,\hat{a}_x + y\sin\theta \cdot \hat{a}_y + z\sin\theta \cdot \hat{a}_z$$

θ is still there, ∴ Use point transformation again.

we know, $\cos\theta = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

∴

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1 - \frac{z^2}{x^2+y^2+z^2}}$$

$$= \sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}$$

∴

$$\vec{A} = x\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}\,\hat{a}_x + y\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}\,\hat{a}_y + z\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}\,\hat{a}_z$$

Ans

# AKTU B.Tech Sem-3

## EE & EE ALLIED

# EMFT

## ( ELECTROMAGNETIC FIELD THEORY )

### Unit-1 : Lec-3

**Topics to be covered**

- Line Integral, Surface Integral & Volume Integral
- Del Operator
- Divergence
- PYQs

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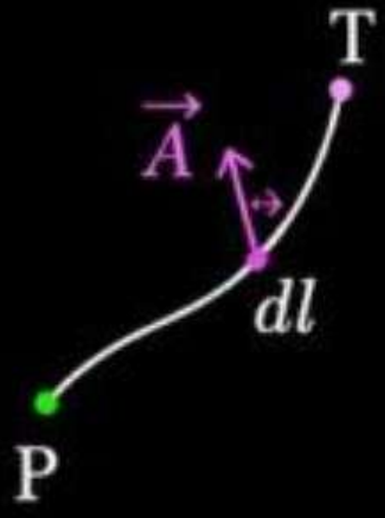
# LINE INTEGRAL : is the integration of a vector along a line ( curve).

( 1 D concept )

↳ means to integrate a quantity along a curve or path.

$$\boxed{\int dl}$$



Line integral of $\vec{A} = \int_P^T \vec{A}\cdot\vec{dl}$

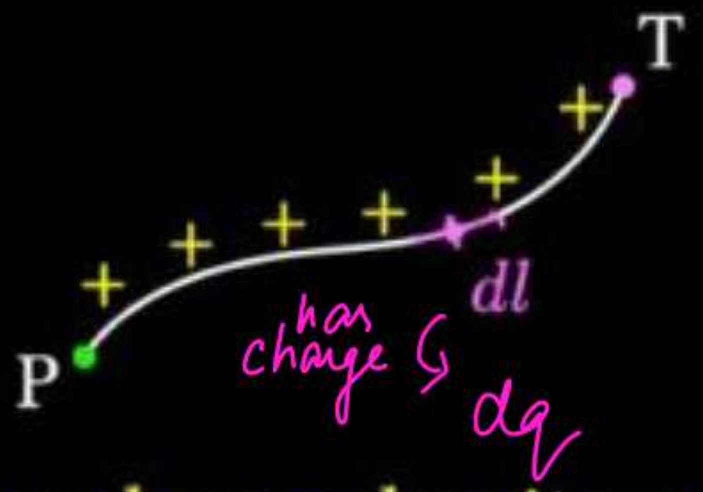
**Example**

$$\boxed{W = \int_P^T \vec{F}\cdot\vec{dl}}$$

$W$ = Work done

$\vec{F}$ = Force

Also, $\oint \vec{F}\cdot\vec{dl} = 0$ Work done in moving a particle along a closed path is zero.



charge exists along a line $\overline{PT}$.

## Line charge density $\rho_L$

$$\rho_L = \frac{\text{charge} \rightarrow dq}{\text{length} \rightarrow dl}$$

charge = $\rho_L$ (length) = $\rho_L$ ($dl$)

$Q = \int dq$

$$\boxed{\text{Total charge } Q = \int \rho_L \, dl}$$

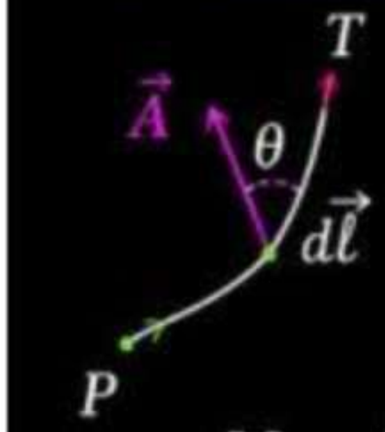
# DIFFERENTIAL LENGTH ELEMENT $(d\vec{l})$

- Is infinitely small vector segment of a path or curve.
- Has both magnitude and direction.

## ① In Cartesian

$(x, y, z)$

$dl$ $(dx\ dy\ dz)$ Learn



$$\boxed{d\vec{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z}$$

Magnitude,

$$\boxed{dl = \sqrt{dx^2 + dy^2 + dz^2}}$$

## ② In Cylindrical

$(\rho, \phi, z)$

Learn → $(d\rho, \rho d\phi, dz)$

$$\boxed{d\vec{l} = d\rho\,\hat{a}_\rho + \rho\, d\phi\,\hat{a}_\phi + dz\,\hat{a}_z}$$

## ③ In Spherical

$(r, \theta, \phi)$

Learn → $(dr, r d\theta, r\sin\theta\, d\phi)$

$$\boxed{d\vec{l} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\, d\phi\,\hat{a}_\phi}$$

### Significance of Differential Length Element

It is essential for calculating line integrals, such as finding the electric potential difference

$$\boxed{\Delta V = -\int \vec{E}\cdot d\vec{l}}$$

or evaluating Ampere's Law for magnetic fields.

Where:

$d\vec{l}$ = Differential length element

$dl$ = Magnitude of $d\vec{l}$

$\hat{a}_x, \hat{a}_y, \hat{a}_z$ = Unit vectors in Cartesian coordinates

$\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ = Unit vectors in Cylindrical coordinates

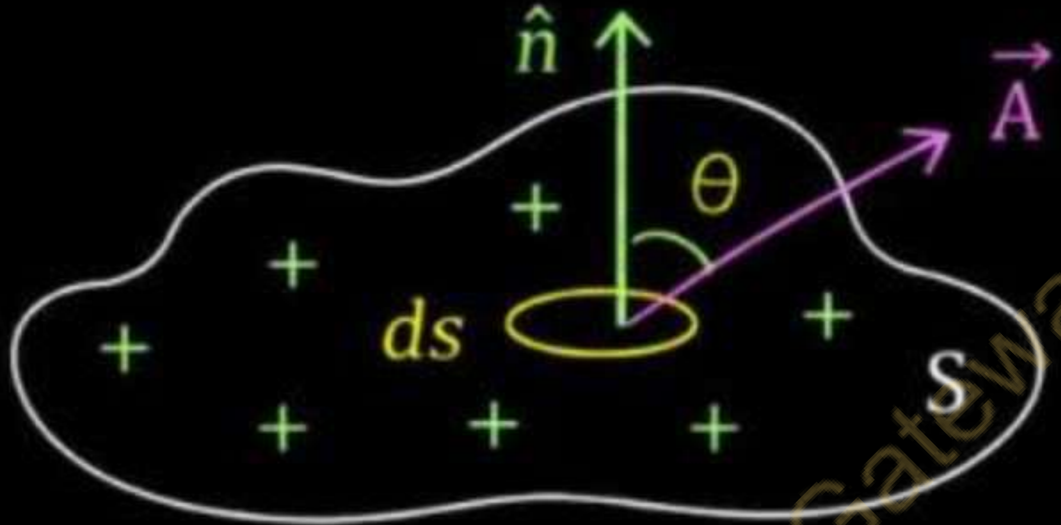
$\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ = Unit vectors in Spherical coordinates

# SURFACE INTEGRAL: is the integration of a vector along a surface.

( 2 D concept )

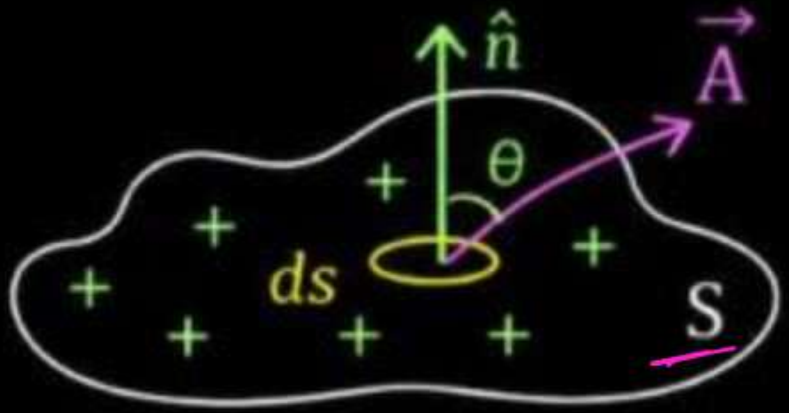
→ means to integrate a quantity over a surface (or area)

$$\iint_S ds \quad or \quad \iint_S dS$$



$$\text{Surface integral of } \vec{A} = \iint \vec{A} \cdot d\vec{S}$$

$$= \int_S \vec{A} \cdot d\vec{S}$$



Surface charge density $\rho_s$

$$\rho_s = \frac{\boxed{\text{charge}}\ dq}{\boxed{\text{area}} \rightarrow ds}$$

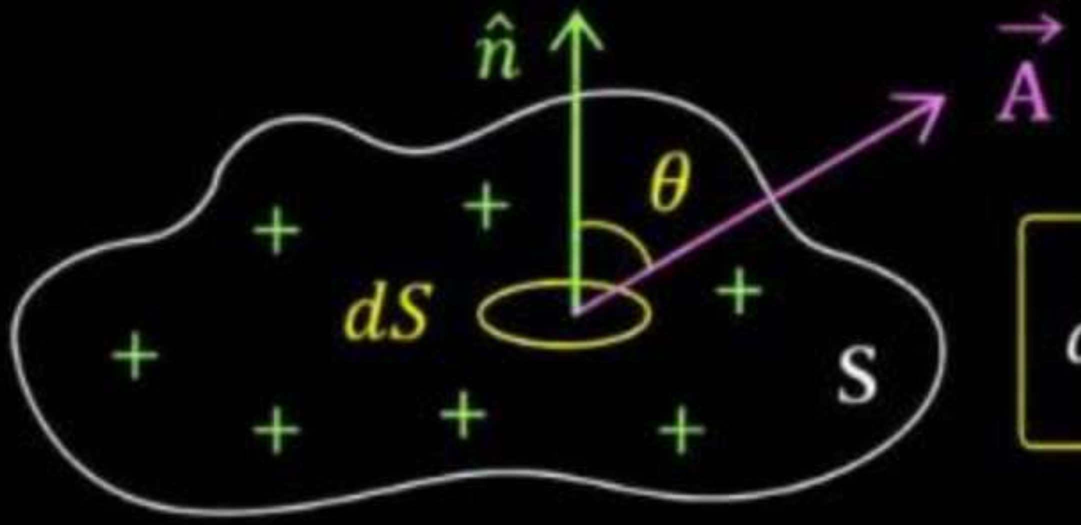
$$\overset{dq}{\text{charge}} = \rho_s\,(Area) = \rho_s\,(ds)$$

$$Q = \iint \boxed{dq}$$

$$\boxed{\text{Total charge } Q = \iint_S \rho_s\, ds}$$

# DIFFERENTIAL AREA ELEMENT ($\vec{dS}$)

Is an infinitesimal element of the surface.



$$\vec{dS} = dS \cdot \hat{n}$$

## ① In Cartesian

dx, dy, dz

$$dS_x = dy\,dz, \quad dS_y = dx\,dz, \quad dS_z = dx\,dy$$

## ② In Cylindrical

dρ, ρdφ, dz

$$(dS)_\rho = \rho\, d\phi\, dz\,,$$
$$(dS)_\phi = d\rho\, dz\,,$$
$$(dS)_z = \rho\, d\rho\, d\phi$$

## ③ In Spherical

dr, r dθ, r sinθ dφ

$$(dS)_r = r^2 \sin\theta\, d\theta\, d\phi$$
$$(dS)_\theta = r \sin\theta\, dr\, d\phi$$
$$(dS)_\phi = r\, dr\, d\theta$$

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## Significance of Differential Area Element

These elements are fundamental for evaluating surface integrals in Gauss's Law, Ampere's Law, and finding magnetic flux.

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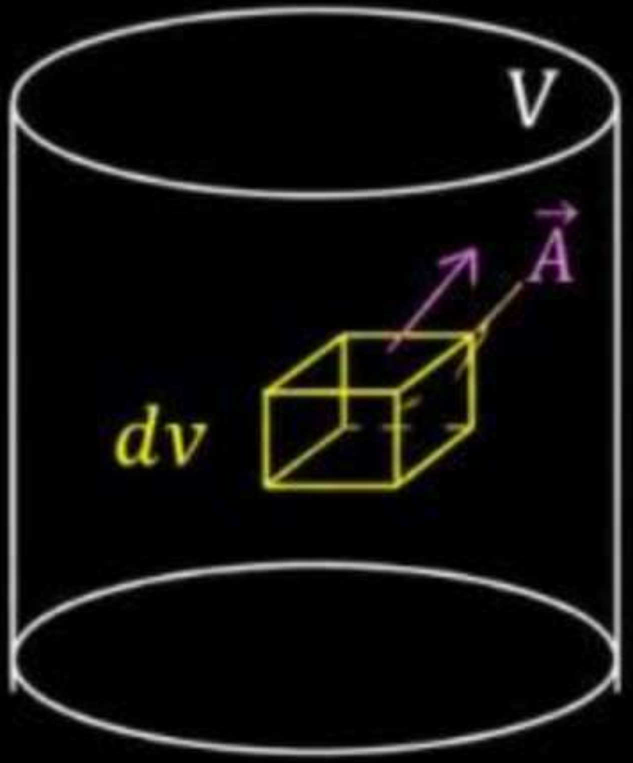
# VOLUME INTEGRAL:- is the integration of a vector along a volume .

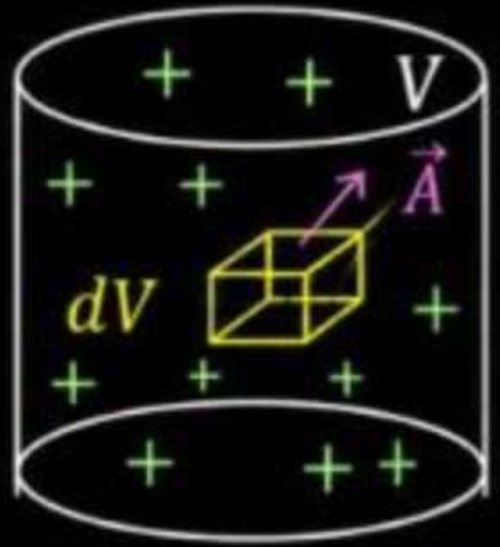
( 3D concept )

$$\boxed{\iiint_V dV \quad \text{or} \quad \int_V . \, dV}$$



Volume integral of $\vec{A} = \iiint \vec{A} \, . \, d\vec{V}$

$$= \int_V \vec{A} \cdot d\vec{V}$$



Volume charge density $\rho_v$

$$\rho_v = \frac{\text{charge} \rightarrow dq}{\text{volume} \rightarrow dV}$$

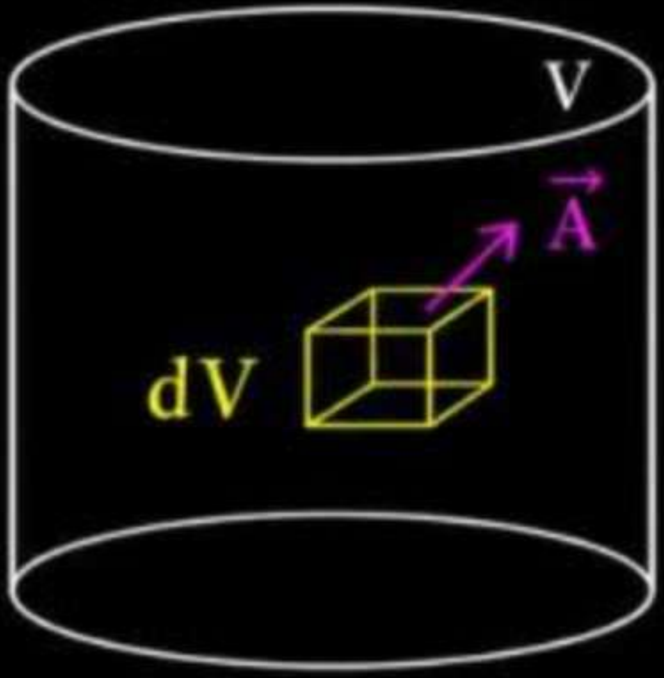
$$\underset{dq}{\text{charge}} = \rho_v(\,\text{volume}) = \rho_v.dV$$

$$Q = \iiint dq$$

$$\boxed{\text{Total charge } Q = \iiint_V \rho_v \; dV}$$

# DIFFERENTIAL VOLUME ELEMENT $dV$

→ infinitesimal element of the volume.



## Significance of Differential Volume Element

Volume Charge Density $\rho_v$: Total charge $\boxed{Q = \iiint_V \rho_v \, dV}$

Energy Density $w$: Total energy $\boxed{W = \iiint_V w \, dV}$

**Field Calculations:** Used in Gauss's Law and Faraday's Law to integrate over a closed or open volume.

1) In cartesian

$dx, dy, dz$

$$dV = dx\,dy\,dz$$

2) In cylindrical

$d\rho, \rho d\phi, dz$

$$dV = \rho\,d\rho\,d\phi\,dz$$

3) In spherical

$dr, r\,d\theta, r\sin\theta\,d\phi$

$$dV = r^2 \sin\theta\,dr\,d\theta\,d\phi$$

# 1. Rectangular (or) Cartesian Co-ordinate System

| Concept | Expression |
|---|---|
| Position of point P | $(x, y, z)$ |
| Unit vectors | $\hat{a}_x, \hat{a}_y, \hat{a}_z$ |
| Position vector | $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$ |
| Differential lengths | $dx, dy, dz$ |
| Differential length (vector form) | $d\vec{l} = dx\, \hat{a}_x + dy\, \hat{a}_y + dz\, \hat{a}_z$ |
| Differential surface | $dS_x = dy\, dz,\ dS_y = dx\, dz,\ dS_z = dx\, dy$ |
| Differential volume | $dV = dx\, dy\, dz$ |

Learn

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# 2. CYLINDRICAL CO-ORDINATE SYSTEM

| Concept | Expression |
|---|---|
| Position of point P | $(\rho, \phi, z)$ |
| Unit vectors | $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ |
| Position vector | $\vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$ |
| Differential lengths | $d\rho, \rho\, d\phi, dz$ |
| Differential length (vector form) | $\vec{dl} = d\rho\, \hat{a}_\rho + \rho\, d\phi\, \hat{a}_\phi + dz\, \hat{a}_z$ |
| Differential Surface | $(dS)_\rho = \rho\, d\phi\, dz$, $(dS)_\phi = d\rho\, dz$, $(dS)_z = \rho\, d\rho\, d\phi$ |
| Differential Volume | $dV = \rho\, d\rho\, d\phi\, dz$ |

Learn

**Q.1) Use cylindrical coordinates to find the area of the curved surface of a right circular cylinder of radius ‘r’ and height ‘h’. Obtain the volume also.**

We Know,

For a right circular cylinder

$$\text{Area} = 2\pi r h$$

$$\text{Volume} = \pi r^2 h$$

These formulae can be derived using the concepts of surface integral and volume integral.

## Derivation :-

### (1) Area

Cylindrical co-ordinate system.

$$\boxed{(dS)_\rho = \rho \, d\phi \, dz}$$

where $\rho = r$ (Given)

$0 \leq \phi \leq 2\pi$

$0 \leq z \leq h$ (Given)

Now, Area $= \iint (dS)_\rho$

$$= \iint \rho \, d\phi \, dz$$

$$= \iint r \, d\phi \, dz \qquad \therefore \; \rho = r$$

$$= r \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{h} dz$$

$$= r \, [\phi]_0^{2\pi} \, [z]_0^{h}$$

$$= r \, [2\pi - 0] \, [h - 0]$$

$$= 2\pi r h$$

$$\boxed{\text{Area} = 2\pi r h}$$

## $(ii)$ Volume

cylindrical coordinate system.

$$\boxed{dV = \rho\, d\rho\, d\phi\, dz}$$

where $\rho = r$ (Given)

$0 \le \phi \le 2\pi$

$0 \le z \le h$ (Given)

Now, Volume = $\iiint dV$

$$= \iiint \rho\, d\rho\, d\phi\, dz$$

Volume $= \int_0^r \rho\, d\rho \int_0^{2\pi} d\phi \int_0^h dz$

$$= \left[\frac{\rho^2}{2}\right]_0^r \left[\phi\right]_0^{2\pi} \left[z\right]_0^h$$

$$= \frac{r^2}{2} \cdot 2\pi \cdot h$$

$$= \pi r^2 h$$

# 3. SPHERICAL CO-ORDINATE SYSTEM

| Concept | Expression |
|---|---|
| Position of point P | $(r, \theta, \phi)$ |
| Unit vectors | $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ |
| Position vector | $\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$ |
| Len Differential lengths | $dr, \quad r\,d\theta, \quad r\sin\theta\,d\phi$ |
| Differential length (vector form) | $\vec{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$ |
| Differential surface | $(dS)_r = r^2 \sin\theta\,d\theta\,d\phi$, $(dS)_\theta = r\sin\theta\,dr\,d\phi$, $(dS)_\phi = r\,dr\,d\theta$ |
| Differential volume | $dV = r^2 \sin\theta\,dr\,d\theta\,d\phi$ |

Learn

| Cartesian | Cylindrical | Spherical |
|---|---|---|
| $dx\,,\ dy\,,\ dz$ | $d\rho\,,\ \rho\,d\phi\,,\ dz$ | $dr\,,\ r\,d\theta\,,\ r\sin\theta\,d\phi$ |
| $dS_x = dy\,dz$<br>$dS_y = dx\,dz$<br>$dS_z = dx\,dy$ | $(dS)_\rho = \rho\,d\phi\,dz$<br>$(dS)_\phi = d\rho\,dz$<br>$(dS)_z = \rho\,d\rho\,d\phi$ | $(dS)_r = r^2\sin\theta\,d\theta\,d\phi$<br>$(dS)_\theta = r\sin\theta\,dr\,d\phi$<br>$(dS)_\phi = r\,dr\,d\theta$ |
| $dV = dx\,dy\,dz$ | $dV = \rho\,d\rho\,d\phi\,dz$ | $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$ |

**Explain all possible forms of surface vector , line vector and volume integral in Spherical system.**

## In Spherical Coordinate System

**Line Vector** $\overrightarrow{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\,d\phi\,\hat{a}_\phi$

**Surface Vector** $(d\vec{S})_r = r^2\sin\theta\,d\theta\,d\phi\,\hat{a}_r$

$(d\vec{S})_\theta = r\sin\theta\,dr\,d\phi\,\hat{a}_\theta$

$(d\vec{S})_\phi = r\,dr\,d\theta\,\hat{a}_\phi$

**Volume Integral** $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

# DEL OPERATOR $\vec{\nabla}$

It is called a **vector differential operator** and is read as " **del** ".

**In Cartesian coordinate system,**

Learn

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{a}_x + \frac{\partial}{\partial y}\,\hat{a}_y + \frac{\partial}{\partial z}\,\hat{a}_z}$$

**where**

$\frac{\partial}{\partial x}$ is partial derivative w.r.t. $x$

$\frac{\partial}{\partial y}$ is partial derivative w.r.t. $y$

$\frac{\partial}{\partial z}$ is partial derivative w.r.t. $z$

eg. $\vec{A} = x^2y\,\hat{a}_x + yz\,\hat{a}_y + z^3x\,\hat{a}_z$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}(x^2y)\,\hat{a}_x\cdot\hat{a}_x + \frac{\partial}{\partial y}(yz)\,\hat{a}_y\cdot\hat{a}_y + \frac{\partial}{\partial z}(z^3x)\,\hat{a}_z\cdot\hat{a}_z$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(z^3x)$$

$$= 2xy + z + 3z^2x$$

$$= 2xy + z + 3z^2x$$

Ans.

# DEL OPERATOR

In cylindrical coordinate system,

Learn

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial p}\,\hat{a}_\rho + \frac{1}{\rho}\,\frac{\partial}{\partial \phi}\,\hat{a}_\phi + \frac{\partial}{\partial z}\,\hat{a}_z}$$

In spherical coordinate system,

Learn

$$\boxed{\vec{\nabla} = \frac{\partial}{\partial r}\,\hat{a}_r + \frac{1}{r}\,\frac{\partial}{\partial \theta}\,\hat{a}_\theta + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \phi}\,\hat{a}_\phi}$$

# DIVERGENCE

**AKTU PYQ-2025-26**

**Explain the divergence of a scalar field.**

(i) Divergence

The dot product of the operator $\overrightarrow{\nabla}$ with a vector $\vec{A}$ is called Divergence of $\vec{A}$.

It is denoted by $\overrightarrow{\nabla}\cdot\vec{A}$ and is written as div $\vec{A}$.

$$\therefore \boxed{\text{div}\,\vec{A} = \overrightarrow{\nabla}\cdot\vec{A}}$$

It is a scalar quantity as it is the dot product of two vectors. ( $\overrightarrow{\nabla}$ and $\vec{A}$ ) .

(ii) Divergence of Scalar

$$\boxed{\overrightarrow{\nabla}\cdot V = \underline{\text{grad}}\ V}$$

where V is scalar.

Gradient

# DIVERGENCE IN DIFFERENT CO-ORDINATE SYSTEMS

In **Cartesian** coordinate system,

Learn

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

In **Cylindrical** coordinate system,

Learn

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}}$$

In **Spherical** coordinate system,

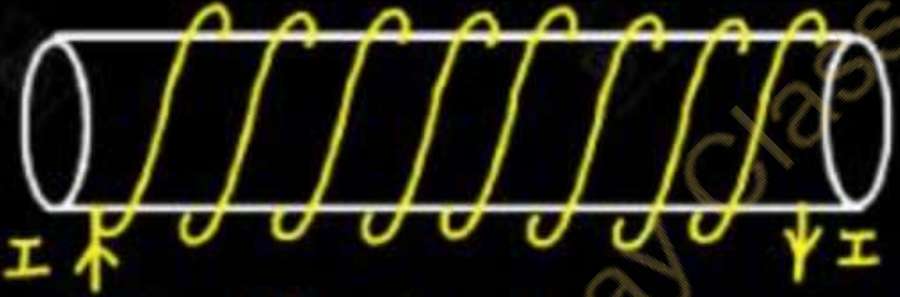
Learn

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}}$$

**Give the physical significance of divergence.**

In electromagnetic field theory (EMFT), the divergence of a vector field ($\vec{\nabla}\cdot\vec{A}$) measures the net outward **flux per unit volume** at a point, representing how much the field "spreads out" (source, positive divergence) or "converges" (sink, negative divergence).

Consider a **solenoid** i.e an electromagnet obtained by winding a coil around a core.



When current passes through it , flux is produced around it.

Such flux completed a closed path through the solenoid , hence the solenoidal field does not diverge.

Mathematically, the vector field having its divergence zero is called a **Solenoidal field.**

$$\boxed{div\,\vec{A} = \vec{\nabla}\cdot\vec{A} = 0}$$

# Questions based on Divergence

**Q.2)** *If* $\vec{A} = y\,\hat{a}_x + (z^2 + y^2)\,\hat{a}_y + (yz + zx)\,\hat{a}_z$ **Then find div $\vec{A}$ at point (1, -2, 3).**

$A_x = y, \quad A_y = (z^2 + y^2), \quad A_z = (yz + zx)$

In cartesian coordinate system

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial (y)}{\partial x} + \frac{\partial (z^2 + y^2)}{\partial y} + \frac{\partial (yz + zx)}{\partial z}$$

$$= 0 + (0 + 2y) + (y + x)$$

$$= x + 3y$$

At point $(1, -2, 3)$

$x = 1, \quad y = -2, \quad z = 3$

$$\vec{\nabla} \cdot \vec{A} = 1 + 3(-2)$$

$$= 1 - 6$$

$$= -5$$

$\boxed{\textbf{Ans.}\ -5}$

## Q.3) Evaluate divergence for $\vec{A}$ = yz $a_x$ + 4xy $a_y$ + y $a_z$ at point (1, -2, 3)

$A_x = yz \ , \quad A_y = 4xy \ , \quad A_z = y$

In cartesian coordinate system

$$\boxed{\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial\,(yz)}{\partial x} + \frac{\partial\,(4xy)}{\partial y} + \frac{\partial\,(y)}{\partial z}$$

$$= 0 + 4x + 0$$

$$= 4x$$

At point (1, –2, 3)

$x = 1 \ , \quad y = -2 \ , \quad z = 3$

$$\vec{\nabla} \cdot \vec{A} = 4\,(1)$$

$$= 4$$

Ans.

## Q.4) If $Q = \rho \sin\phi \, a_\rho + \rho^2 z \, a_\phi + z \cos\phi \, a_z$, determine divergence of Q.

$$Q_\rho = \rho \sin\phi$$

$$Q_\phi = \rho^2 z$$

$$Q_z = z \cos\phi$$

Divergence $\vec{Q}$ or $\vec{\nabla} \cdot \vec{Q}$ in cylindrical coordinate system

$$\boxed{\vec{\nabla} \cdot \vec{Q} = \underbrace{\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho Q_\rho)}_{I^{st} \text{ term}} + \underbrace{\frac{1}{\rho} \frac{\partial Q_\phi}{\partial \phi}}_{II^{nd} \text{ term}} + \underbrace{\frac{\partial Q_z}{\partial z}}_{III^{rd} \text{ term}}}$$

$I^{st}$ term

$$\frac{1}{\rho} \cdot \frac{\partial}{\partial \rho} \left( \rho \cdot \rho \sin\phi \right)$$

$$= \frac{1}{\rho} \cdot \frac{\partial}{\partial \rho} \left( \rho^2 \sin\phi \right)$$

$$= \frac{1}{\rho} \left( 2\rho \sin\phi \right) = 2 \sin\phi$$

$II^{nd}$ term

$$\frac{1}{\rho} \cdot \frac{\partial}{\partial \phi} \left( \rho^2 z \right) = 0$$

$III^{rd}$ term

$$\frac{\partial}{\partial z} \left( z \cos\phi \right) = \cos\phi \cdot \frac{\partial}{\partial z}(z) = \cos\phi (1) = \cos\phi$$

$$\therefore \quad \nabla \cdot \vec{Q} = (2 \sin\phi) + 0 + (\cos\phi) = 2 \sin\phi + \cos\phi$$

$$\boxed{\vec{\nabla} \cdot \vec{Q} = 2 \sin\phi + \cos\phi} \quad \text{— (1)}$$

# Q.5) Find divergence in cylindrical coordinates

**Given:** $\vec{A} = 2\rho\cos^2\phi\,\widehat{a_\rho} + 3\rho^2\sin z\,\widehat{a_\phi} + 4z\sin^2\phi\,\widehat{a_z}$, **Find:** $\nabla\cdot\vec{A}$

$$A_\rho = 2\rho\cos^2\phi$$

$$A_\phi = 3\rho^2\sin z$$

$$A_z = 4z\sin^2\phi$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \underbrace{\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)}_{I} + \underbrace{\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}}_{II} + \underbrace{\frac{\partial A_z}{\partial z}}_{III}}$$

$$(I)\quad \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot 2\rho\cos^2\phi)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(2\rho^2\cos^2\phi)$$

$$= \frac{1}{\rho}(4\rho\cos^2\phi) = 4\cos^2\phi$$

$$(II)\quad \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} = \frac{1}{\rho}\frac{\partial}{\partial\phi}(3\rho^2\sin z)$$

$$= \frac{1}{\rho}(0) = 0$$

$$(III)\quad \frac{\partial A_z}{\partial z} = \frac{\partial}{\partial z}(4z\sin^2\phi)$$

$$= 4\sin^2\phi\,\frac{\partial}{\partial z}(z)$$

$$= 4\sin^2\phi$$

$$\therefore\ \vec{\nabla}\cdot\vec{A} = 4\cos^2\phi + 0 + 4\sin^2\phi$$

$$= 4(\cos^2\phi + \sin^2\phi)$$

$$= 4(1) = 4$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = 4}$$ **Ans.**

**Q.6) If $\vec{A} = 10\,\hat{a}_r + 5\sin\theta\,\widehat{a_\theta}$, then find $\overline{\nabla}\cdot\vec{A}$.**

$$A_r = 10$$
$$A_\theta = 5\sin\theta$$
$$A_\phi = 0$$

In spherical coordinate system,

$$\boxed{\vec{\nabla}\cdot\vec{A} = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right)}_{I} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right)}_{II} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}}_{III}}$$

$$(I)\quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot 10\right)$$
$$= \frac{10}{r^2}\frac{\partial}{\partial r}\left(r^2\right)$$
$$= \frac{10}{r^2}\cdot 2r = \frac{20}{r} \quad ①$$

$$(II)\quad \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(5\sin\theta\cdot\sin\theta\right)$$
$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(5\sin^2\theta\right)$$
$$= \frac{1}{r\sin\theta}\left(5\cdot 2\sin\theta\cos\theta\right)$$
$$= \frac{10\cos\theta}{r} \quad ②$$

$$(III)\quad \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(0)$$
$$= 0 \quad ③$$

$$\therefore\ \overline{\nabla}\cdot\vec{A} = (1) + (2) + (3) = \frac{20}{r} + \frac{10\cos\theta}{r} + 0$$

$$\boxed{\overline{\nabla}\cdot\vec{A} = \frac{20 + 10\cos\theta}{r}}$$

## Q.7) Find the value of constant *c* for which the vector $\vec{A} = (x+3y)\,\widehat{a_x} + (y-2z)\,\widehat{a_y} + (x+cz)\,\widehat{a_z}$ is solenoidal.

For $\vec{A}$ to be solenoidal,

$$\boxed{\vec{\nabla}\cdot\vec{A} = 0}$$

In cartesian coordinate system

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

$$A_x = (x+3y)$$

$$A_y = (y-2z)$$

$$A_z = (x+cz)$$

$$\therefore\ \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+cz) = 0$$

$$\therefore\ (1+0) + (1-0) + (0+c) = 0$$

$$1 + 1 + c = 0$$

$$2 + c = 0$$

$$\boxed{c = -2}$$

**Ans.**

# AKTU B.Tech Sem-3

## EE & EE ALLIED

# EMFT

## ( ELECTROMAGNETIC FIELD THEORY )

### Unit-1 : Lec-4

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**Topics to be covered**

- Curl
- Gradient
- PYQs

**By Swati Ma'am**

- M.Tech (Honours)
- B.Tech (Honours)
- GATE Qualified

# CURL

**CURL :-** is the circulation of a vector field around a closed path . $(\vec{\nabla} \times \vec{A})$

$$\boxed{curl\ \vec{A} = rot\ \vec{A} = \vec{\nabla} \times \vec{A}}$$

→ Vector quantity

### In cartesian

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

### In cylindrical

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho . \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

### In spherical

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\,\hat{a}_\theta & r \sin\theta\ \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin\theta\, A_\phi \end{vmatrix}$$

# PHYSICAL SIGNIFICANCE OF CURL

1) When a rigid body is in motion, then curl of its linear velocity at any point gives twice its angular velocity in magnitude and direction .

$$\boxed{\text{Curl } \vec{v} = 2\,\vec{\omega}}$$

2) Ampere's circuital law $\boxed{\vec{\nabla} \times \vec{H} = \vec{J}}$

3) Faraday's Law $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

4) If curl of a vector field exists then the field is called rotational.

For **Irrotational** vector field, the curl vanishes i.e **curl is zero.**

**Q.1** If $\vec{P} = x^2yz\,\hat{a}_x + xz\,\hat{a}_z$. Determine curl of $\vec{P}$

$\vec{P}$ is in cartesian coordinate system,

$$P_x = x^2yz \;, \quad P_y = 0 \;, \quad P_z = xz$$

**In cartesian**

$$\vec{\nabla} \times \vec{P} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P_x & P_y & P_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{P} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & 0 & xz \end{vmatrix}$$

$$\vec{\nabla} \times \vec{P} = \hat{a}_x \left[ \frac{\partial}{\partial y}(xz) - \frac{\partial}{\partial z}(0) \right]$$

$$- \hat{a}_y \left[ \frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial z}(x^2yz) \right]$$

$$+ \hat{a}_z \left[ \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(x^2yz) \right]$$

$$= \hat{a}_x\,[0] - \hat{a}_y \left[ z - x^2y \right]$$

$$+ \hat{a}_z \left[ -x^2z \right]$$

$$= \left[ x^2y - z \right] \hat{a}_y - (x^2z)\,\hat{a}_z$$ Ans.

## Q.2 Evaluate curl for $\vec{P} = yz\,\hat{a}_x + 4xy\,\hat{a}_y + y\,\hat{a}_z$ at point (1, - 2, 3)

$\vec{P}$ *is* in *cartesian* coordinate system,

$P_x = yz\,,\quad P_y = 4xy\,,\quad P_z = y$

In cartesian

$$\vec{\nabla} \times \vec{P} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P_x & P_y & P_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{P} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 4xy & y \end{vmatrix}$$

$$\vec{\nabla} \times \vec{P} = \hat{a}_x\left[\frac{\partial\,(y)}{\partial\,y} - \frac{\partial\,(4xy)}{\partial\,z}\right]$$

$$-\,\hat{a}_y\left[\frac{\partial\,(y)}{\partial\,x} - \frac{\partial\,(y\,z)}{\partial\,z}\right]$$

$$+\,\hat{a}_z\left[\frac{\partial\,(4\,x\,y)}{\partial\,x} - \frac{\partial\,(y\,z)}{\partial\,y}\right]$$

$$= \hat{a}_x\,[1 - 0] - \hat{a}_y\,[0 - y] + \hat{a}_z\,[4\,y - z]$$

$$= \hat{a}_x + y\,\hat{a}_y + (4\,y - z)\,\hat{a}_z$$

At Point $(1, -2, 3)$ , $x = 1\,,\; y = -2\,,\; z = 3$

$$\vec{\nabla} \times \vec{P} = \hat{a}_x - 2\,\hat{a}_y + [4\,(-2) - 3]\,\hat{a}_z$$

$$= \hat{a}_x - 2\,\hat{a}_y - 11\,\hat{a}_z$$ *Ans*

**Q. 3. Prove curl $\vec{A} = 0$, if $\vec{A} = (yz\,\hat{a}_x + zx\,\hat{a}_y + xy\,\hat{a}_z)$.**

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$\vec{A}$ is in Cartesian co-ordinate system.

$$\vec{A} = A_x\,\hat{a}_x + A_y\,\hat{a}_y + A_z\,\hat{a}_z$$

where $A_x = yz,\ A_y = zx,\ A_z = xy$

**In Cartesian,**

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{a}_x \left[\frac{\partial (xy)}{\partial y} - \frac{\partial (zx)}{\partial z}\right] - \hat{a}_y \left[\frac{\partial (xy)}{\partial x} - \frac{\partial (yz)}{\partial z}\right]$$

$$+ \hat{a}_z \left[\frac{\partial (zx)}{\partial x} - \frac{\partial (yz)}{\partial y}\right]$$

$$= \hat{a}_x\,[x - x] - \hat{a}_y\,[y - y] + \hat{a}_z\,[z - z]$$

$$= 0$$

**Hence Proved.**

**Q.4** Given a vector field $\vec{A} = \rho \sin\phi \, \hat{a}_\rho + \rho^2 \, \hat{a}_\phi$ in cylindrical coordinates, calculate the curl of $\vec{A}$ and evaluate it at point $( 2, \frac{\pi}{2}, 0)$.

$\therefore$ $\vec{A}$ is given in cylindrical co-ordinates,

$$\vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$$

$$\therefore \quad A_\rho = \rho \sin\phi, \quad A_\phi = \rho^2, \quad A_z = 0$$

$$\therefore \quad \vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho \sin\phi & \rho . \rho^2 & 0 \end{vmatrix}$$

$$= \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho \sin\phi & \rho^3 & 0 \end{vmatrix}$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho \sin\phi & \rho^3 & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[ \hat{a}_\rho \, (0-0) - \rho \hat{a}_\phi \, (0-0) + \hat{a}_z \left(3\rho^2 - \rho \cos\phi\right)\right]$$

$$= \frac{1}{\rho} \cdot \bar{\rho} \left[ 3\rho - \cos\phi \right] \hat{a}_z$$

$$= \left[ 3\rho - \cos\phi \right] \hat{a}_z$$

At $\left(2, \frac{\pi}{2}, 0\right)$, i.e. $\rho = 2$, $\phi = \frac{\pi}{2}$, $z = 0$,

$$\vec{\nabla} \times \vec{A} = \left[3(2) - \cos\left(\frac{\pi}{2}\right)\right] \hat{a}_z$$

$$= 6\hat{a}_z$$ Ans

**Q.** 5. Given that a vector function $\vec{A} = (3x + c_1 z)\,\hat{a}_x + (c_2 x - 5z)\,\hat{a}_y + (4x - c_3 y + c_4 z)\,\hat{a}_z$. Examine the values $c_1, c_2, c_3$ and $c_4$ if $\vec{A}$ is irrotational and solenoidal.

$$\vec{A} \text{ is irrotational} \downarrow \Rightarrow \vec{\nabla} \times \vec{A} = 0$$

$$\vec{A} \text{ is solenoidal} \downarrow \Rightarrow \vec{\nabla} \cdot \vec{A} = 0$$

(i) $\vec{\nabla} \times \vec{A} = 0$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = 0$$

$$\vec{\nabla} \cdot \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x + c_1 z & c_2 x - 5z & 4x - c_3 y + c_4 z \end{vmatrix} = 0$$

$$\hat{a}_x(-c_3 + 5) - \hat{a}_y(4 - c_1) + \hat{a}_z(c_2) = 0$$

$$(5-c_3)\,\hat{a}_x-(4-c_1)\,\hat{a}_y+c_2\,\hat{a}_z=0$$

So, each component must be zero,

$$5-c_3=0 \quad\Rightarrow\quad \boxed{c_3=5}$$

$$-(4-c_1)=0 \quad\Rightarrow\quad \boxed{c_1=4}$$

$$c_2=0 \quad\Rightarrow\quad \boxed{c_2=0}$$

(ii) $\vec{\nabla}\cdot\vec{A}=0$

$\therefore$

$$\frac{\partial}{\partial x}(3x+c_1z)+\frac{\partial}{\partial y}(c_2x-5z)+\frac{\partial}{\partial z}(4x-c_3y+c_4z)=0$$

$$3+c_4=0 \quad\Rightarrow\quad \boxed{c_4=-3}$$

Substitute values of $c_1, c_2, c_3, c_4$ in $\vec{A}$.

$$\vec{A}=(3x+4z)\,\hat{a}_x-5z\,\hat{a}_y+(4x-5y-3z)\,\hat{a}_z$$

Ans

# GRADIENT

**GRADIENT :-** is the dot product of a scalar and del operator.

$$\boxed{grad\ \ V = \overrightarrow{\nabla} \cdot V}$$

→ where **V** is a scalar function.

→ is a vector quantity.

### In cartesian coordinate system

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\,\hat{a}_x + \frac{\partial V}{\partial y}\,\hat{a}_y + \frac{\partial V}{\partial z}\,\hat{a}_z$$

### In cylindrical coordinate system

$$\vec{\nabla} V = \frac{\partial V}{\partial \rho}\,\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\,\hat{a}_\phi + \frac{\partial V}{\partial z}\,\hat{a}_z$$

### In spherical coordinate system

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\,\hat{a}_r + \frac{1}{r}\,\frac{\partial V}{\partial \theta}\,\hat{a}_\theta + \frac{1}{r\sin\theta}\,\frac{\partial V}{\partial \phi}\,\hat{a}_\phi$$

# PHYSICAL SIGNIFICANCE OF GRADIENT

1. The gradient is a differential operator by means of which we can associate a **vector field** with a **scalar field.**

   **For eg:**

$$\boxed{\vec{E} = -\,\text{grad V}}$$

   where **E** is Electric Field Intensity , **V** is Potential ( Scalar Quantity )

   Negative Sign ⇒ Direction of electric field intensity is **opposite** to the direction of **increase** of potential.

2. If P represents pressure (scalar quantity) , then

$$\boxed{\vec{\nabla}\ P}$$

   represents **Pressure Gradient** (Vector Quantity) which defines rate of change of pressure with distance.

**Q.6** If $V = e^{-z} \sin 2x \cosh y$, determine Gradient of V.

V is in cartesian coordinate system

$$\boxed{grad\ V = \vec{\nabla} V}$$

$$\boxed{\vec{\nabla} V = \underbrace{\frac{\partial V}{\partial x}\hat{a}_x}_{I} + \underbrace{\frac{\partial V}{\partial y}\hat{a}_y}_{II} + \underbrace{\frac{\partial V}{\partial z}\hat{a}_z}_{III}}$$

(I) $\frac{\partial V}{\partial x}\hat{a}_x = \frac{\partial}{\partial x}\left(e^{-z} \sin 2x \cosh y\right)\hat{a}_x$

$= e^{-z} \cosh y \,(2\cos 2x)\,\hat{a}_x$

$= 2e^{-z} \cos 2x \cosh y \,\hat{a}_x$ —— ①

(II) $\frac{\partial V}{\partial y}\hat{a}_y = \frac{\partial}{\partial y}\left(e^{-z} \sin 2x \cosh y\right)\hat{a}_y$

$= e^{-z} \sin 2x \sinh y \,\hat{a}_y$ —— ②

(III) $\frac{\partial V}{\partial z}\hat{a}_z = \frac{\partial}{\partial z}\left(e^{-z} \sin 2x \cosh y\right)\hat{a}_z$

$= \sin 2x \cosh y \frac{\partial}{\partial z}\left(e^{-z}\right)\hat{a}_z$

$= \sin 2x \cosh y \left(-e^{-z}\right)\hat{a}_z$

$= -e^{-z} \sin 2x \cosh y \,\hat{a}_z$ —— ③

$\therefore \vec{\nabla} V = 2e^{-z} \cos 2x \cosh y \,\hat{a}_x$

$+ e^{-z} \sin 2x \sinh y \,\hat{a}_y - e^{-z} \sin 2x \cosh y \,\hat{a}_z$

**Ans.**

**Q.7** For the scalar field $\Phi(x, y) = x^2 \sin 5y$ , calculate gradient of $\Phi$ .

$\Phi(x, y)$ is given in Cartesian. Let $\Phi(x, y) = V$

$$\text{grad } \Phi = \text{grad } V = \vec{\nabla} V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

$$= \frac{\partial}{\partial x}(x^2 \sin 5y)\,\hat{a}_x + \frac{\partial}{\partial y}(x^2 \sin 5y)\,\hat{a}_y + \frac{\partial}{\partial z}(x^2 \sin 5y)\,\hat{a}_z$$

$$= 2x \sin 5y\,\hat{a}_x + x^2(5 \cos 5y)\,\hat{a}_y + 0$$

$$= \left[2x \sin 5y\right]\hat{a}_x + \left[5x^2 \cos 5y\right]\hat{a}_y \quad \underline{\textit{Ans.}}$$

**Q.8** Given $V = 4x^3y^2 - 3xyz$ . Determine E at ( 3, -4, 5).

$$\vec{E} = -\vec{\nabla} V$$

Compute $\vec{\nabla} \cdot V$

$$\vec{\nabla} \cdot V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

$$= \frac{\partial}{\partial x}\left(4x^3y^2 - 3xyz\right)\hat{a}_x + \frac{\partial}{\partial y}\left(4x^3y^2 - 3xyz\right)\hat{a}_y + \frac{\partial}{\partial z}\left(4x^3y^2 - 3xyz\right)\hat{a}_z$$

$$= \left(12x^2y^2 - 3yz\right)\hat{a}_x + \left(8x^3y - 3xz\right)\hat{a}_y + \left(-3xy\right)\hat{a}_z$$

**Substitute** $(x, y, z) = (3, -4, 5)$ in

$$\vec{\nabla} \cdot V = (12x^2y^2 - 3yz)\,\hat{a}_x + (8x^3y - 3xz)\,\hat{a}_y + (-3xy)\,\hat{a}_z$$

$$= [12(3)^2(-4)^2 - 3(-4)(5)]\,\hat{a}_x + [8(3)^3(-4) - 3(3)(5)]\,\hat{a}_y + [-3(3)(-4)]\,\hat{a}_z$$

$$= 1788\,\hat{a}_x - 909\,\hat{a}_y + 36\,\hat{a}_z$$

$$\therefore \quad \vec{E} = -\vec{\nabla} V$$

$$\vec{E} = -[1788\,\hat{a}_x - 909\,\hat{a}_y + 36\,\hat{a}_z]$$ **Ans**

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**Q.** 9 Compute $\vec{\nabla} \times \vec{\nabla} V$, if $V = xyz$.

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

$$= \frac{\partial (xyz)}{\partial x}\hat{a}_x + \frac{\partial (xyz)}{\partial y}\hat{a}_y + \frac{\partial (xyz)}{\partial z}\hat{a}_z$$

$$= yz\,\hat{a}_x + xz\,\hat{a}_y + xy\,\hat{a}_z$$

Now,

$$\vec{\nabla} \times \vec{\nabla} V = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \hat{a}_x\left[\frac{\partial (xy)}{\partial y} - \frac{\partial (xz)}{\partial z}\right] - \hat{a}_y\left[\frac{\partial (xy)}{\partial x} - \frac{\partial (yz)}{\partial z}\right]$$

$$+ \hat{a}_z\left[\frac{\partial (xz)}{\partial x} - \frac{\partial (yz)}{\partial y}\right]$$

$$= \hat{a}_x\,[x - x] - \hat{a}_y\,[y - y] + \hat{a}_z\,[z - z]$$

$$= 0$$ **Ans.**

# AKTU B.Tech Sem-3

## EE & EE ALLIED

# EMFT

## ( ELECTROMAGNETIC FIELD THEORY )

### Unit-1 : Lec-5

**Topics to be covered**

- Laplacian of a Scalar
- Scalar Potential
- Stoke's Theorem
- Divergence Theorem
- PYQs

GATEWAY CLASSES

**By Swati Ma'am**

- M.Tech (Honours)
- B.Tech (Honours)
- GATE Qualified

# LAPLACIAN OF A SCALAR $(\nabla^2)$

Laplacian is a differential operator given by the divergence of the gradient of a scalar function.

$$\nabla^2 V = \vec{\nabla} \cdot \vec{\nabla} V$$

**In Cartesian ,**

$$\left\{ \begin{aligned} &\vec{\nabla} V = \frac{\partial V}{\partial x}\,\hat{a}_x + \frac{\partial V}{\partial y}\,\hat{a}_y + \frac{\partial V}{\partial z}\,\hat{a}_z \\ &\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla} V = \left[\frac{\partial}{\partial x}\,\hat{a}_x + \frac{\partial}{\partial y}\,\hat{a}_y + \frac{\partial}{\partial z}\,\hat{a}_z\right]\left[\frac{\partial V}{\partial x}\,\hat{a}_x + \frac{\partial V}{\partial y}\,\hat{a}_y + \frac{\partial V}{\partial z}\,\hat{a}_z\right] \\ &\boxed{\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla} V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}} \end{aligned} \right.$$

# LAPLACIAN OF A SCALAR

In Cylindrical,

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}}$$

In Spherical,

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2}}$$

## SIGNIFICANCE OF LAPLACIAN OF A SCALAR

If $\nabla^2 V = 0$ , then scalar field is harmonic in given region i.e of the form of sine or cosine.

## 1. Derive the expression for Laplacian of a scalar in all three coordinates.

$$\nabla^2 V = \vec{\nabla}\cdot(\vec{\nabla} V)$$

**(1) Cartesian coordinates $(x, y, z)$**

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z \quad \text{and} \quad \vec{\nabla} = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

$$\vec{\nabla}\cdot\vec{\nabla} V = \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial V}{\partial x}\right)\hat{a}_x\cdot\hat{a}_x}_{} + \frac{\partial}{\partial y}\underbrace{\left(\frac{\partial V}{\partial y}\right)\hat{a}_y\cdot\hat{a}_y}_{} + \frac{\partial}{\partial z}\underbrace{\left(\frac{\partial V}{\partial z}\right)\hat{a}_z\cdot\hat{a}_z}_{}$$

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}$$

## (ii) Cylindrical coordinates $(\rho, \phi, z)$

$$\vec{\nabla} \cdot V = \frac{\partial V}{\partial \rho}\,\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\,\hat{a}_\phi + \frac{\partial V}{\partial z}\,\hat{a}_z$$

**Divergence Formula**

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

Here $\vec{A} = \vec{\nabla} \cdot V$

$$\therefore \quad \nabla^2 V = \vec{\nabla} \cdot \vec{\nabla} V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial}{\partial \varphi}\underbrace{\left(\frac{1}{\rho}\frac{\partial V}{\partial \phi}\right)} + \frac{\partial}{\partial z}\underbrace{\left(\frac{\partial V}{\partial z}\right)}$$

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \varphi^2} + \frac{\partial^2 V}{\partial z^2}}$$

# (iii) Spherical co-ordinates $(r, \theta, \phi)$

$$\vec{\nabla} \cdot V = \frac{\partial V}{\partial r}\,\hat{a}_r + \frac{1}{r}\,\frac{\partial V}{\partial \theta}\,\hat{a}_\theta + \frac{1}{r\sin\theta}\,\frac{\partial V}{\partial \phi}\,\hat{a}_\phi$$

## Divergence Formula

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\,\frac{\partial A_\phi}{\partial \phi}$$

Here $\vec{A} = \vec{\nabla} \cdot V$

$$\therefore\ \nabla^2 V = \vec{\nabla} \cdot \vec{\nabla} V = \frac{1}{r^2}\,\frac{\partial}{\partial r}\underbrace{\left(r^2 \frac{\partial V}{\partial r}\right)}_{} + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \theta}\underbrace{\left(\frac{1}{r}\,\frac{\partial V}{\partial \theta}\sin\theta\right)}_{} + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial \phi}\underbrace{\left(\frac{1}{r\sin\theta}\,\frac{\partial V}{\partial \phi}\right)}_{}$$

$$\boxed{\nabla^2 V = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 V}{\partial \phi^2}}$$

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**2. Find the Laplacian if $V = \rho^2 z \cos 2\phi$**

Laplacian operator in cylindrical coordinates

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}}$$

**1st term** $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\partial}{\partial \rho}\rho^2 z \cos 2\phi\right)$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(2\rho^2 z \cos 2\phi\right)$$

$$= \frac{1}{\rho}\left(2 \cdot 2\rho z \cos 2\phi\right)$$

$$= \frac{1}{\rho}\left(4\rho z \cos 2\phi\right)$$

$$= 4 z \cos 2\phi$$

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$II^{nd}$ term $\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\left(\rho^2 z \cos 2\phi\right)$

$$= \frac{1}{\rho^2}\,\rho^2 z\,\frac{\partial^2}{\partial \phi^2}\left(\cos 2\phi\right)$$

$$= z\,\frac{\partial}{\partial \phi}\left(-2\sin 2\phi\right)$$

$$= -4z\cos 2\phi$$

$III^{rd}$ term $\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left(\rho^2 z \cos 2\phi\right)$

$$= \rho^2 \cos 2\phi\,\frac{\partial^2}{\partial z^2}\,z$$

$$= \rho^2 \cos 2\phi\,\frac{\partial}{\partial z}\,(1)$$

$$= 0$$

$\therefore$

$\nabla^2 V =$

$4z\cos 2\phi$

$-4z\cos 2\phi$

$+0$

$\therefore\ \nabla^2 V = 0$

Ans

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# OPERATORS OF VECTOR CALCULUS

| S.No. | Operation | Expression | Symbol/Name | Input Type | Output Type |
|---|---|---|---|---|---|
| 1 | Divergence | $\vec{\nabla} \cdot \vec{A}$ | div | Vector | Scalar |
| 2 | Gradient | $\vec{\nabla} \vec{V}$ | grad | Scalar | Vector |
| 3 | Curl | $\vec{\nabla} \times \vec{A}$ | curl or rot | Vector | Vector |
| 4 | Laplacian | $\vec{\nabla}^{2} V$ | - | Scalar | Scalar |

# Types of Fields

| S.No. | Type | Condition | Description |
|---|---|---|---|
| 1 | Solenoidal Field | $\vec{\nabla} \cdot \vec{A} = 0$ | Field has zero divergence ( no net flux). |
| 2 | Irrotational Field | $\vec{\nabla} \times \vec{A} = 0$ | Field has zero curl ( no rotation). |
| 3 | Harmonic Function | $\nabla^2 V = 0$ | Scalar potential satisfies Laplace's equation. |

# SCALAR POTENTIAL

$\int ① \, dx$

$\frac{\partial \phi}{\partial y}$

$\int$

## How to find Scalar Potential $\Phi(x,y)$

Que)

**Given:** $F(x,y) = \langle 2x, 3y^2 \rangle$

**So,**

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = 2x & ① \\ \dfrac{\partial \phi}{\partial y} = 3y^2 & ② \end{cases}$$

**Step 1:** Integrate ① w.r.t $x$ to obtain $\Phi$

$$\phi = \int \frac{\partial \phi}{\partial x}\, dx = \int 2x\, dx = x^2 + \underline{f(y)} \quad ③$$

Here, $f(y)$ is an arbitrary function of $y$.

To find $f(y)$ :-

**Step 2:** Differentiate $\Phi$ w.r.t $y$

From ③

$$\frac{\partial \phi}{\partial y} = f'(y) \quad - ④$$

But from ②

$$\boxed{f'(y) = 3y^2}$$

**Integrate**

$$f(y) = \int 3y^2\, dy = y^3 + C \quad - ⑤$$

**Final Potential**

Substitute ⑤ into ③

$$\boxed{\Phi(x,y) = x^2 + y^3 + C}$$

**Answer**

# SCALAR POTENTIAL

$\int ① \, dx$

$\frac{\partial \phi}{\partial y}$

$\int$

**Ques.** Find Scalar Potential of
$F(x,y,z) = (2xy\ ,\ x^2 + 2yz\ ,\ y^2)$

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = 2xy & —① \\ \dfrac{\partial \phi}{\partial y} = x^2 + 2yz & —② \\ \dfrac{\partial \phi}{\partial z} = y^2 & —③ \end{cases}$$

**Integrate ① w.r.t x :**

$$\phi = \int 2xy\, dx = x^2 y + f(y,z) \quad —④$$

**Differentiate ④ w.r.t y :**

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y,z) \quad —⑤$$

**Compare ② with ⑤ :**

$$x^2 + f'(y,z) = x^2 + 2yz$$

$$f'(y,z) = 2yz$$

**Integrate**

$$\boxed{f(y,z) = y^2 z + C} \quad —⑥$$

**Put ⑥ in ④ :**

$$\boxed{\Phi(x,y,z) = x^2 y + y^2 z + C}$$

**Ans.**

**4.** Show that the vector field $\mathbf{F} = yz^2\,\hat{\imath} + (xz^2+2)\,\hat{\jmath} + (2xyz-1)\,\hat{k}$ is conservative by finding a scalar potential.

$$\vec{F} = yz^2\,\hat{\imath} + (xz^2+2)\,\hat{\jmath} + (2xyz-1)\,\hat{k}$$

is conservative if and only if

$$\vec{\nabla} \times \vec{F} = \vec{0}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & xz^2+2 & 2xyz-1 \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(2xyz-1) - \frac{\partial}{\partial z}(xz^2+2)\right]$$

$$-\hat{\jmath}\left[\frac{\partial}{\partial x}(2xyz-1) - \frac{\partial}{\partial z}(yz^2)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(xz^2+2) - \frac{\partial}{\partial y}(yz^2)\right]$$

$$= \hat{\imath}\,[\,2xz - 0 - 2xz - 0\,]$$

$$-\hat{\jmath}\,[\,2yz - 0 - 2yz\,]$$

$$+\hat{k}\,[\,z^2 + 0 - z^2\,]$$

$$= 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = \vec{0}$$

$\therefore$ $\vec{F}$ is conservative.

Now, to find its scalar potential,

$$\vec{F} = yz^2\,\hat{i} + (xz^2 + 2)\,\hat{j} + (2xyz - 1)\,\hat{k}$$

$$\left\{\begin{aligned} \frac{\partial \phi}{\partial x} &= yz^2 && ① \\ \frac{\partial \phi}{\partial y} &= xz^2 + 2 && ② \\ \frac{\partial \phi}{\partial z} &= 2xyz - 1 && ③ \end{aligned}\right.$$

Integrate ①,

$$\phi = \int yz^2\,dx = xyz^2 + f(y,z) \quad - ④$$

$$\frac{\partial \phi}{\partial y} = xz^2 + f'(y,z) \quad - ⑤$$

② = ⑤

$$xz^2 + f'(y,z) = xz^2 + 2$$

$$\left\{\begin{array}{l} \int ① \\ \dfrac{\partial \phi}{\partial y} \\ \int \end{array}\right.$$

$$f'(y,z) = 2$$

Integrate w.r.t. $y$

$$f(y,z) = 2y + C \quad - ⑥$$

Put ⑥ in ④

$$\boxed{\phi = xyz^2 + 2y + C}$$

# STOKE'S THEOREM

**AKTU PYQ-2022-23, 2021-22, 2019-20**

$\oint \rightarrow \iint$

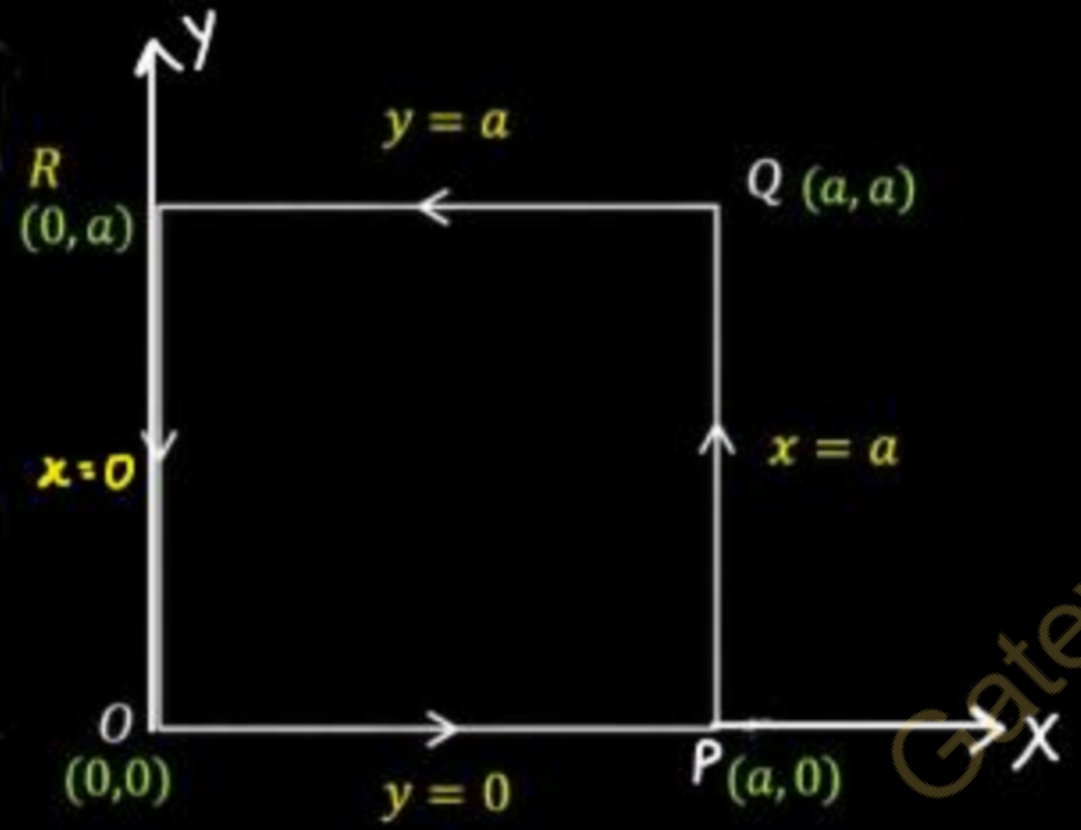
## State and prove Stoke's Theorem.

It states that the line integral of a vector field $\vec{\mathrm{A}}$ around a closed curve is equal to the surface integral of the curl $\vec{\mathrm{A}}$ taken over the surface S surrounded by the closed curve i.e.,

$$\boxed{\oint_C \vec{A}.d\vec{l} = \iint_S (\text{curl}\,\vec{A}).d\vec{S} = \iint_S (\vec{\nabla}\times\vec{A}).d\vec{S}}$$

**Verify Stoke's theorem for the vector $\vec{A} = x(x\,\hat{a}_x + y\,\hat{a}_y)$ integrated round the square, in the plane z = 0, whose sides are along the lines x = 0, y = 0, x = a, y = a.**

$$\boxed{\oint_C \vec{A}.d\vec{l} = \iint_S \text{curl}\,\vec{A}).\vec{dS} = \iint_S (\vec{\nabla}\times\vec{A}) . \vec{dS}}$$



Here, LHS is

$$\oint_C \vec{A}.d\vec{l} = \int_{OP}\vec{A}.d\vec{l} + \int_{PQ}\vec{A}.d\vec{l} + \int_{QR}\vec{A}.d\vec{l} + \int_{RO}\vec{A}.d\vec{l}$$

**OP** $$\int_{OP}\vec{A}.d\vec{l} = \int_0^a (x^2\,\hat{a}_x + xy\,\hat{a}_y)\cdot(dx\,\hat{a}_x)$$

$$= \int_0^a x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \quad -①$$

**PQ** $$\int_{PQ}\vec{A}.d\vec{l} = \int_0^a (x^2\,\hat{a}_x + xy\,\hat{a}_y)\cdot(dy\,\hat{a}_y)$$

$$= \int_0^a xy\,dy = x\left[\frac{y^2}{2}\right]_0^a = \frac{xa^2}{2} = \frac{a^3}{2}. \quad ②$$

$\because$ P→Q
x = a

**QR** $$\int_{QR}\vec{A}.d\vec{l} = \int_a^0 (x^2\,\hat{a}_x + xy\,\hat{a}_y)\cdot(dx\,\hat{a}_x)$$

$$= \int_a^0 x^2\,dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3} \quad -③$$

**RO** $$\int_{RO}\vec{A}.d\vec{l} = \int_a^0 (x^2\,\hat{a}_x + xy\,\hat{a}_y)\cdot(dy\,\hat{a}_y)$$

$$= \int_a^0 xy\,dy = x\left[\frac{y^2}{2}\right]_a^0 = -\frac{xa^2}{2} = 0 \quad ④$$

$\because$ R→O
x = 0

From (1) + (2) + (3) + (4),

$$\oint_C \vec{A}.d\vec{l} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} - 0 = \frac{a^3}{2}$$

**Further,**

$$RHS = \iint_S (\text{curl } \vec{A}) \cdot d\vec{S} = \iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} \quad -⑤$$

$$\vec{A} = ( x^2 \hat{a}_x + xy\, \hat{a}_y )$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \hat{a}_x \left[ \frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(xy) \right] - \hat{a}_y \left[ \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2) \right]$$

$$+ \hat{a}_z \left[ \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2) \right]$$

$$= \underline{y\, \hat{a}}_z$$

**Now,** $d\vec{S} = dx\ dy\ \underline{\hat{a}_z}$

**Put in ⑤,**

$$RHS = \iint_S (y\, \hat{a}_z) \cdot (dx\ dy\ \hat{a}_z)$$

$$= \iint_S y\ dx\ dy \qquad \therefore \hat{a}_z \cdot \hat{a}_z = 1$$

$$= \int_{\substack{0 \\ x=0}}^{a} dx \int_{\substack{0 \\ y=0}}^{a} y\ dy$$

$$= \int_0^a dx \left[ \frac{y^2}{2} \right]_0^a$$

$$= \int_0^a dx \left( \frac{a^2}{2} \right) = \left[ x \right]_0^a \cdot \frac{a^2}{2}$$

$$= a \cdot \frac{a^2}{2} = \frac{a^3}{2}$$

$LHS = RHS$

**Hence Proved.**

# DIVERGENCE THEOREM

- **Gauss Theorem of Divergence** $\iint \longrightarrow \iiint$

It states that the surface integral of the normal component of vector $\vec{A}$ taken over a closed surface $S$ is equal to the volume integral of the divergence of vector $\vec{A}$ over the volume $V$ enclosed by the surface $S$, i.e

$$\boxed{\iint_S \vec{A} \cdot d\vec{S} = \iiint_V div\,\vec{A}\;dV = \iiint_V (\vec{\nabla} \cdot \vec{A})\,dV}$$

# QUESTIONS BASED ON DIVERGENCE THEOREM

**Que.** Use the divergence theorem to show that $\boxed{\iint_S \vec{F}\cdot\hat{n}\;dS = \frac{12}{5}\pi r^5}$

**where** S is the surface of sphere of radius r and

$$\boxed{\vec{F} = x^3\hat{a}_x + y^3\hat{a}_y + z^3\hat{a}_z} \longrightarrow$$

is in Cartesian co-ordinate system but surface is spherical. *

**Sol.** According to Gauss Divergence Theorem

$$\iint_S \underbrace{\vec{F}\cdot\hat{n}\;dS}_{\hat{n}dS} = \iiint_V (\vec{\nabla}\cdot\vec{F})\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)\right] dV$$

$$= \iiint_V (3x^2 + 3y^2 + 3z^2)\,dV$$

$$= \iiint_V (3x^2 + 3y^2 + 3z^2)\, dV$$

$$= \iiint_V 3(x^2 + y^2 + z^2)\, dV$$

$$= \iiint_V \underline{3r^2}\, dV$$

$\ast$ $\boxed{\because\ x^2 + y^2 + z^2 = r^2}$

$\therefore$ Since S is a sphere (given)

$$\boxed{dV = r^2 \sin\theta\, d\theta\, d\phi\, dr}$$

$$= \iiint_V 3r^2 (r^2 \sin\theta\, d\theta\, d\phi\, dr)$$

$$= 3\int_0^r r^4\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= 3\left[\frac{r^5}{5}\right]_0^r \left[-\cos\theta\right]_0^\pi \left[\phi\right]_0^{2\pi}$$

$$= 3\left[\frac{r^5}{5}\right]_0^r \left[-\cos\pi + \cos 0\right]\left[2\pi - 0\right]$$

$$= 3\left[\frac{r^5}{5}\right]_0^r \left[-(-1) + 1\right]\left[2\pi\right]$$

$$= 3\left(\frac{r^5}{5} - 0\right)\left[2\right]\left[2\pi\right]$$

$\therefore\ \cos\pi = -1$

$\cos 0 = 1$

$$= \frac{12}{5}\pi r^5$$

Hence Proved.

**Que** – Given $\vec{A} = r^2\,\hat{a}_r + \sin\theta\,\hat{a}_\phi$, determine the net outward flux through the closed hemispherical surface defined by

$0 \le r \le 5,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le 2\pi$

Flux is to be calculated

By Divergence Theorem, $\boxed{\iint_S \vec{A}\cdot d\vec{S} = \iiint_V \mathrm{div}\,\vec{A}\,dV = \iiint_V (\vec{\nabla}\cdot\vec{A})\,dV}$

Flux = $\iiint_V (\vec{\nabla}\cdot\vec{A})\,dV$ — ①

$\vec{A}$ is in spherical coordinate system $(r, \theta, \phi)$.

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}}$$

$$\boxed{A_r = r^2,\ A_\theta = 0,\ A_\phi = \sin\theta}$$

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## To find $\iiint (\vec{\nabla} \cdot \vec{A})\, dV$

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So,

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta\, A_\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} (A_\phi)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \cdot r^2) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \cdot 0) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} (\sin\theta)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} (r^4) + 0 + 0$$

$$= \frac{1}{r^2} (4 r^3)$$

$$= 4 r$$

$$\boxed{\therefore\ \vec{\nabla} \cdot \vec{A} = 4 r} \quad — ②$$

In Spherical Coordinate System, $\boxed{dV = r^2 \sin\theta \, dr \, d\theta \, d\phi}$ — ③

Put (2) and (3) in (1)

$$\text{Flux} = \iiint_V 4r \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 4 \int_0^5 r^3 \, dr \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

∴ Hemispherical surface

$\therefore \; \theta = 0 \text{ to } \frac{\pi}{2}$

$$= 4 \left[\int_0^5 r^3 \, dr\right] \left[\int_0^{\pi/2} \sin\theta \, d\theta\right] \left[\int_0^{2\pi} d\phi\right]$$

$$= 4 \left[\frac{r^4}{4}\right]_0^5 \left[-\cos\theta\right]_0^{\pi/2} \left[\phi\right]_0^{2\pi}$$

$$= (5^4)\left(-\cos\frac{\pi}{2} + \cos 0\right)(2\pi)$$

$$= (625)(-0 - (-1))(2\pi)$$

$$= (625)(1)(2\pi)$$

$$= 1250\pi \quad \text{Ans.}$$

**5. Prove that divergence of a curl of a vector is zero , using Stoke's Theorem.**

To prove $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

**Proof :** Acc. to Divergence Theorem,

$$\iiint_V \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})\,dV = \iint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S}$$

Apply Stoke's Theorem,

$$\iint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \oint_C \vec{A}\cdot d\vec{\ell}$$

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Consider a closed surface,

For a closed surface (like a sphere),

the boundary C shrinks to a single point

$$\therefore \quad \iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} = 0$$

$$\therefore \quad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$$

**Hence Proved.**

AKTU PYQ-2020-21

**Que-** Verify the divergence theorem for the

$$\vec{D} = \rho^2 \cos 2\Phi \, \hat{a}_\rho + z \sin \Phi \, \hat{a}_\Phi$$

over the closed surface of the cylinder

$$0 \leq z \leq 1 \,,\; \rho \leq 4$$

Divergence Theorem states

$$\boxed{\iint_S \vec{A} \cdot d\vec{S} = \iiint_V (\vec{\nabla} \cdot \vec{A})\, dV}$$

Consider here $\vec{D}$ as $\vec{A}$

$\vec{D}$ is in cylindrical coordinate system ( $\rho$ , $\Phi$ , $z$ )

## RHS : Compute divergence

$$\boxed{\vec{\nabla} \cdot \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(D_\phi) + \frac{\partial}{\partial z}(D_z)}$$

Here, $D_\rho = \rho^2 \cos 2\Phi \quad , \quad D_\phi = z \sin \Phi \quad , \quad D_z = 0$

So, $$\vec{\nabla} \cdot \vec{\mathrm{D}} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, \rho^2 \cos 2\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(z \sin \phi) + \frac{\partial}{\partial z}(0)$$

$$= \frac{1}{\rho} \cos 2\phi \frac{\partial}{\partial \rho}(\rho^3) + \frac{1}{\rho} z \frac{\partial}{\partial \phi}(\sin \phi)$$

$$= \frac{1}{\rho} \cos 2\phi \, (3\rho^2) + \frac{1}{\rho} z \, (\cos \phi)$$

$$= 3\rho \cos 2\phi + \frac{z \cos \phi}{\rho} \quad — ①$$

# Volume Integral

$$\iiint_V (\vec{\nabla}\cdot\vec{D})\,dV = \int_0^{2\pi}\int_0^{1}\int_0^{4}\left(3\rho\ \cos 2\phi + \frac{z\cos\phi}{\rho}\right)\underbrace{\rho\,d\rho\,d\phi\,dz}_{dV}$$

$\phi = 0$ to $2\pi$, $\rho = 0$ to $1$, $z = 0$ to $4$

# Continuation of Volume Integral

$$\int_{\phi=0}^{2\pi}\int_{z=0}^{1}\int_{\rho=0}^{4}\left(3\rho\ \cos 2\phi + \frac{z\cos\phi}{\rho}\right)\rho\,d\rho\,d\phi\,dz$$

**Term 1**

$$\int_{\phi=0}^{2\pi}\int_{z=0}^{1}\int_{\rho=0}^{4} 3\rho^2\cos 2\phi\,d\rho\,d\phi\,dz$$

$$= 3\int_0^1 dz\int_0^4 \rho^2\,d\rho\int_0^{2\pi}\cos 2\phi\,d\phi$$

$$= 3\,[z]_0^1\left[\frac{\rho^3}{3}\right]_0^4\left[\frac{\sin 2\phi}{2}\right]_0^{2\pi}$$

$$= 3(1)\left(\frac{64}{3}\right)\left(\frac{\sin 4\pi - \sin 0}{2}\right)$$

$$= 64(0) = 0$$

$$= 0$$

$\because\ \sin 4\pi = 0$

$\sin 0 = 0$

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**Term 2**

$$\int_{\phi=0}^{2\pi}\int_{z=0}^{1}\int_{\rho=0}^{4} z\cos\Phi \; d\rho\, d\Phi\, dz$$

$$= \int_0^1 z\,dz \int_0^4 d\rho \int_0^{2\pi} \cos\Phi\, d\Phi$$

$$= \left[\frac{z^2}{2}\right]_0^1 \left[\rho\right]_0^4 \left[\sin\Phi\right]_0^{2\pi}$$

$$= \left(\frac{1}{2}-\frac{0^2}{2}\right)(4-0)\,(\sin 2\pi - \sin 0)$$

$$= \left(\frac{1}{2}\right)(4)\,(\sin 2\pi - \sin 0)$$

$$= 2(0)$$

$$= 0$$

$\therefore\ \sin 2\pi = 0$

$\sin 0 = 0$

**Therefore**

$$\iiint_V (\vec{\nabla}\cdot\vec{D})\, dV = 0$$

## LHS : Surface Integral

$$\iint_S \vec{D} \cdot d\vec{S} = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} \rho^2 \cos 2\phi \, \hat{a}_\rho + z \sin\phi \, \hat{a}_\phi) \cdot \big( \underbrace{\rho \, d\phi \, dz \ \hat{a}_\rho}_{dS_\rho} \big)$$

$$= \int_0^{2\pi}\int_0^1 \rho^3 \cos 2\phi \quad d\phi \ dz \ + \ 0 \ \Rightarrow \ \hat{a}_\phi \cdot \hat{a}_\rho = 0$$

$$= \rho^3 \int_0^{2\pi} \cos 2\phi \, d\phi \ \cdot \ \int_0^1 dz$$

$$= \rho^3 \big[\sin 2\phi\big]_0^{2\pi} \ \big[\, z \,\big]_0^1$$

$$= \rho^3 \left[ \sin 2\phi \right]_0^{2\pi} (1)$$

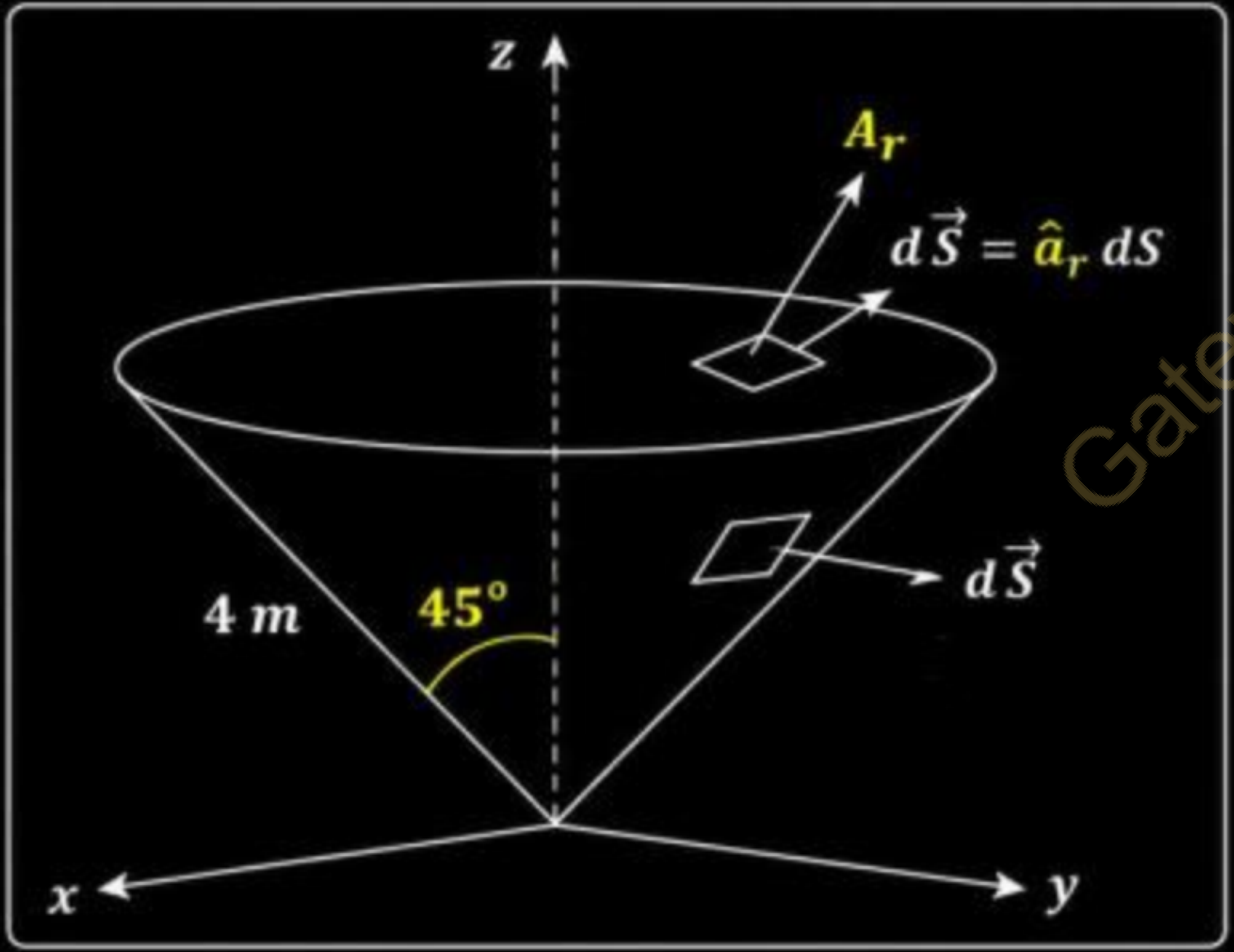
$$= \rho^3 \left( \sin 2\pi - \sin 0 \right) (1)$$

$$= 0$$

$\because \ \sin 2\pi = 0$
$\sin 0 = 0$

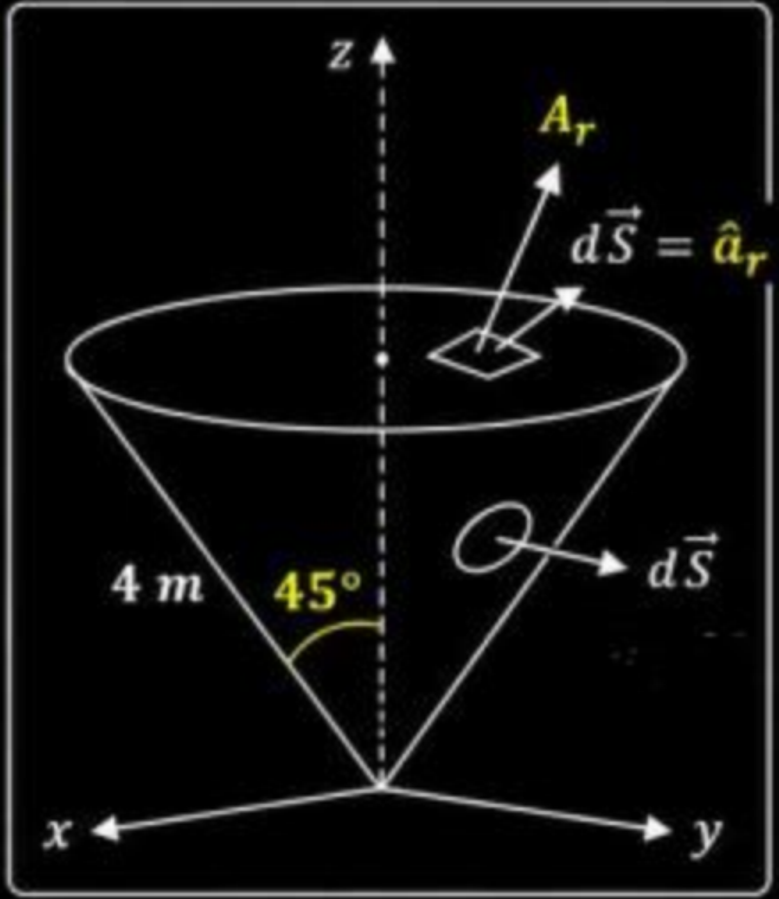
**LHS = RHS ∴ Divergence Theorem Verified.**

**Que** - Given that $\vec{A} = \left(\frac{5r^2}{4}\right)\hat{a}_r$ is in spherical coordinates, solve **both sides of** the divergence theorem the volume enclosed by $r = 4$ m, and $\theta = \frac{\pi}{4}$, **shown in below figure.**


z
$A_r$
$d\vec{S} = \hat{a}_r\, dS$
4 m
45°
$d\vec{S}$
x
y

## Divergence Theorem

$$\boxed{\iint_S \vec{A}\cdot d\vec{S} = \iiint_V div\,\vec{A}\; dV = \iiint_V (\vec{\nabla}\cdot\vec{A})\, dV}$$



$\vec{A}$ is in Spherical Coordinate system (Given)

$$\boxed{A_r = \frac{5r^2}{4}\;,\quad A_\theta = 0\,,\quad A_\phi = 0}$$

**LHS**

$$\iint_S \vec{A}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/4} A_r\, dS_r = \int_0^{2\pi}\int_0 \frac{5r^2}{4}\, r^2 \sin\theta\, d\theta\, d\phi$$

$\rightarrow dS_r = r^2 \sin\theta\, d\theta\, d\phi$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{5r^4}{4} \sin\theta\, d\theta\, d\phi$$

$$= \frac{5r^4}{4}\int_0^{\pi/4} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= -\frac{5r^4}{4}\left[\cos\theta\right]_0^{\pi/4}\left[\phi\right]_0^{2\pi}$$

$$= -\frac{5}{4}(4^4)\left(\cos\frac{\pi}{4} - \cos 0\right)(2\pi) \qquad \because r = 4 \text{ given}$$

$$= 320\,(2\pi)\left(1 - \frac{1}{\sqrt{2}}\right)$$

**RHS :**

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(A_\phi\right)}$$

$$A_r = \frac{5r^2}{4}\ ,\quad A_\theta = 0\ ,\quad A_\phi = 0$$

$$\iiint_V (\vec{\nabla}\cdot\vec{A})\, dV = \iiint_V \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{5r^2}{4}\right) dV$$

$$\boxed{dV = r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

$$\iiint_V (\vec{\nabla}\cdot\vec{A})\, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{5r^2}{4}\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$\phi$ : 0 to $2\pi$, $\theta$ : 0 to $\pi/4$, $r$ : 0 to 4

$$\iiint_V (\vec{\nabla}\cdot\vec{A})\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} \frac{1}{r^2}\frac{\partial}{\underline{\partial r}}\left(r^2\cdot\frac{5r^2}{4}\right) r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} \frac{1}{r^2}\left(\frac{5}{4}\times 4r^3\right) r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} 5r^3\,dr\,\sin\theta\,d\theta\,d\phi$$

$$= 5\int_0^{4} r^3\,dr\int_0^{\pi/4}\sin\theta\,d\theta\int_0^{2\pi}d\phi$$

$$= 5\left[\frac{r^4}{4}\right]_0^4\left[-\cos\theta\right]_0^{\pi/4}\left[\phi\right]_0^{2\pi}$$

$$= -\frac{5}{4}\,(4)^4\left[\cos\frac{\pi}{4}-\cos 0\right]\left[2\pi\right]$$

$$= -5(64)\left[\frac{1}{\sqrt{2}}-1\right](2\pi)$$

$$= 320\left[1-\frac{1}{\sqrt{2}}\right](2\pi)$$

$$= 320\,(2\pi)\left[1-\frac{1}{\sqrt{2}}\right]$$

= LHS

$$\boxed{\text{LHS} = \text{RHS}}$$

Hence Proved.

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